FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2008

The problem set is available from Friday October 17. The set consists of 2 problems written on 4 pages. For solving the problems, it may be useful to consult the relevant sections of the lecture notes.

Deadline for returning solutions
Friday October 24. *Written/printed* solutions should be returned to Ekspedisjonskontoret in the Physics Building.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471, email: j.m.leinaas@fys.uio.no) or the assistant Mads Stormo Nilsson (Office: room 340V, email: m.s.nilsson@fys.uio.no).

Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Spin splitting in positronium

Positronium is a bound system of an electron and a positron. The two particles have the same mass m and charges of opposite signs $\pm e$, with e denoting the electron charge. The energy spectrum of the bound system is similar to that of a hydrogen atom, but the energy scale is different since the *reduced mass* of the two-particle system has about half the value in positronium compared to hydrogen. Positronium has a finite life time since the electron and the positron will eventually annihilate.

The ground state of positronium is degenerate due to the spin degrees of freedom of the two particles. We distinguish between *para-positronium*, which is a spin *singlet* state with total spin $S = 0$, and *ortho-positronium* which is a *triplet* state with total spin $S = 1$. Para-positronium has a life time of 125 picoseconds while the life time of ortho-positronium is about 140 nanoseconds.

The interaction between the magnetic moments of the two particles give rise to a (hyperfine) splitting of the ground state energy, so that the singlet state has a slightly lower energy than the triplet state. In the following we make the simplifying assumption that this effect can be studied in the four-dimensional spin space of the two particles. This means that we assume no coupling between the spin and orbital coordinates of the particles so that the wave function of the orbital motion is the same for all the spin states and can therefore be neglected.

We denote in the following the *spin up* state vector of the z-component of the spin for any of the two particles as $|+\rangle$ and the *spin down* state by $|-\rangle$. The four dimensional space of spin states has the tensor product form $\mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_p$, with $\mathcal{H}_e$ as the two-dimensional spin space of the electron and $\mathcal{H}_p$ as the spin space of the positron. The full space is spanned by the four product states

$$
\begin{aligned}
|++\rangle &= |+\rangle\otimes|+\rangle\,, \quad |+-\rangle = |+\rangle\otimes|-\rangle\,, \\
|-+\rangle &= |-\rangle\otimes|+\rangle\,, \quad |--\rangle = |-\rangle\otimes|-\rangle\,,
\end{aligned} \tag{1}
$$

where we assume the first factor in the tensor product to describe the *electron* spin. In the four-dimensional spin space the spin operators of the electron and the positron have the following forms,

$$
\begin{aligned}
\hat{\mathbf{S}}_e &= \frac{\hbar}{2}\boldsymbol{\sigma}_e\otimes\mathbb{1}_p \equiv \frac{\hbar}{2}\boldsymbol{\Sigma}_e \\
\hat{\mathbf{S}}_p &= \frac{\hbar}{2}\mathbb{1}_e\otimes\boldsymbol{\sigma}_p \equiv \frac{\hbar}{2}\boldsymbol{\Sigma}_p
\end{aligned} \tag{2}
$$

with $\mathbb{1}_e$ as the identity operator in the two-dimensional spin space of the electron,$\mathbb{1}_p$ as the identity operator in the spin space of the positron, and $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_p$ as the Pauli matrices acting in the two-dimensional spin spaces of the electron and the positron respectively.

a) Show that in the product basis we have

$$
\langle ij|\boldsymbol{\Sigma}_e\cdot\boldsymbol{\Sigma}_p|kl\rangle = \langle i|\boldsymbol{\sigma}_e|k\rangle\cdot\langle j|\boldsymbol{\sigma}_p|l\rangle \tag{3}
$$

b) Find the operator product $\hat{\mathbf{S}}_e\cdot\hat{\mathbf{S}}_p$ expressed as a 4×4 matrix in the product basis. (In the matrix representation list the basis vectors in the order $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$.)

We now introduce another basis, the *spin basis* with the four vectors

$$
|0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \tag{4}
$$

and

$$
\begin{aligned}
|1,1\rangle &= |++\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\
|1,-1\rangle &= |--\rangle
\end{aligned} \tag{5}
$$

c) Show that $\hat{\mathbf{S}}_e\cdot\hat{\mathbf{S}}_p$ is a diagonal matrix in the new basis.

The total (intrinsic) spin of the two particles is $\hat{\mathbf{S}} = \hat{\mathbf{S}}_e + \hat{\mathbf{S}}_p$. Show that the new basis vectors are eigenstates of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ and find the eigenvalues. Check that the result for the eigenvalues is consistent with (4) being the singlet state and (5) being the triplet state.

The Hamiltonian in the spin space can be written in the form

$$
H_0 = E_0\mathbb{1} + \kappa\hat{\mathbf{S}}_e\cdot\hat{\mathbf{S}}_p \tag{6}
$$

where E_0 is the ground state energy with spin effects excluded, $\mathbb{1}$ is the identity operator in the four-dimensional spin space and κ is a positive constant determined by the magnetic moments of the particles.

A magnetic field $\mathbf{B}$ is turned on in the z direction. This leads to a splitting of the spin energy states, referred to as the *Zeeman effect*. The form of the modified Hamiltonian is

$$\hat{H} = E_0 \mathbb{1} + \kappa \hat{\mathbf{S}}_e \cdot \hat{\mathbf{S}}_p + \lambda\hbar(\hat{S}_{e\,z} - \hat{S}_{p\,z}) \tag{7}$$

with λ as a parameter proportional to B.

d) Write the Hamilton H as a 4×4 matrix in the spin basis.

e) Solve the eigenvalue problem for the Hamiltonian (7) and find the energies expressed in terms of the parameters E_0, κ and λ. Plot the energies as functions of $x \equiv \lambda/\kappa$ for fixed E_0 and κ.

Two of the energy eigenstates are mixtures of $|0, 0\rangle$ and $|1, 0\rangle$. We write these two states as

$$\begin{aligned} |A\rangle &= a\,|+-\rangle + b\,|-+\rangle \\ |B\rangle &= -b^*\,|+-\rangle + a^*\,|-+\rangle \end{aligned} \tag{8}$$

where a and b are functions of x, with $|a|^2 + |b|^2 = 1$.

f) Give the expressions for the corresponding density operators $\hat{\rho}_A$ and $\hat{\rho}_B$, and for the *reduced density operators* $\hat{\rho}_{Ae}, \hat{\rho}_{Ap}$ and $\hat{\rho}_{Be}, \hat{\rho}_{Bp}$ of the electron and positron subsystems. The degree of entanglement in the system is given by the *von Neumann entropy* of the reduced density operators. Show that the degree of entanglement is the same for $|A\rangle$ and $|B\rangle$ and can be expressed as a function of $|a|^2$.

g) Determine the function $|a(x)|^2$ from the eigenvalue problem in e) and use this to make a plot of the degree of entanglement in the system as a function of x for the two states $|A\rangle$ and $|B\rangle$. Are these *maximum entanglement states* for any value of x?

2 Driven harmonic oscillator

The Hamiltonian of a one-dimensional harmonic oscillator is given by the expression

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega_0^2\hat{x}^2) = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{9}$$

with the *raising* and *lowering* operators defined by

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0\hat{x} + i\hat{p})\ , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0\hat{x} - i\hat{p}) \tag{10}$$

The time evolution operator is

$$\hat{\mathcal{U}}_0(t) = e^{-\frac{i}{\hbar}t\hat{H}_0} \tag{11}$$

The coherent states of the oscillator are defined as eigenvectors of the lowering operator

$$\hat{a}|z\rangle = z|z\rangle \tag{12}$$

The general coherent state $|z\rangle$ is related to the ground state of the oscillator $|0\rangle$ by

$$|z\rangle = \hat{\mathcal{D}}(z)|0\rangle = e^{-z^*z}e^{z\hat{a}^\dagger}|0\rangle \tag{13}$$

where the unitary shift operator is given by

$$\hat{\mathcal{D}}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}} \tag{14}$$

a) Show that for a general operator $\hat{A}$ we have the relation

$$\hat{\mathcal{U}} e^{\hat{A}} \hat{\mathcal{U}}^{-1} = e^{\hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^{-1}} \tag{15}$$

and use that to calculate the operator $\hat{\mathcal{U}}_0(t)\hat{\mathcal{D}}(z)\hat{\mathcal{U}}_0(t)^\dagger$. Make use of the result to determine the time dependent state vector $|\psi(t)\rangle$, when this initially is a coherent state $|\psi(0)\rangle = |z_0\rangle$. Show that $|\psi(t)\rangle$ at later times t is also a coherent state.

We next assume the harmonic oscillator to be under influence of a time dependent external potential, so that the hamiltonian now is

$$\hat{H} = \hat{H}_0 + \hat{W}(x,t) \tag{16}$$

In the following we assume the external potential to have the specific form

$$\hat{W}(x,t) = A\hat{x}\sin\omega t \tag{17}$$

with A as a constant and ω as the oscillation frequency of the external potential.

b) Find the Heisenberg equation of motion for $\hat{x}$ and $\hat{p}$ and show that they correspond to the equation of motion of a *driven* harmonic oscillator, that is subject to the periodic force $f(t) = A\sin\omega t$.

c) Give the definition of the time evolution operator $\hat{\mathcal{U}}_I(t)$ in the *interaction picture* and show that it satisfies an equation of the form

$$i\hbar\frac{d}{dt}\hat{\mathcal{U}}_I(t) = \hat{H}_I(t)\hat{\mathcal{U}}_I(t) \tag{18}$$

Assume $\hat{W}$ is treated as the interaction. Show that $\hat{H}_I(t)$ then is a linear function of $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{H}_I(t) = \theta(t)^*\,\hat{a} + \theta(t)\,\hat{a}^\dagger \tag{19}$$

and determine the function $\theta(t)$.

d) Show that the equation (18) has a solution of the form

$$\hat{\mathcal{U}}_I(t) = e^{\xi(t)\hat{a}^\dagger - \xi^*(t)\hat{a}}\, e^{i\phi(t)} \tag{20}$$

with $\xi(t)$ as a complex and $\phi(t)$ as a real function of time. What are the equations that these two functions should satisfy?

e) Use the expressions for $\hat{\mathcal{U}}_0(t)$ and $\hat{\mathcal{U}}_I(t)$ to find the time dependent state vector $|\psi(t)\rangle$ in the Schrödinger picture, with the same initial condition as in a). Show that also in this case it describes a time dependent coherent state, of the form $|\psi(t)\rangle = e^{i\gamma(t)}|z(t)\rangle$. Find $z(t)$ expressed in terms of z_0, $\xi(t)$ and ω_0.

f) Determine the function $\xi(t)$ and find an explicit expression for $z(t)$. The corresponding real coordinate is $x(t) = \sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,z(t)$. Does this coordinate satisfy the classical equation of motion of the driven harmonic oscillator?