

FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2008

Problem set 3

3.1 Spin operators and Pauli matrices

A spin half operator \hat{S} is defined in the standard way as

$$\hat{S} = \frac{\hbar}{2} \boldsymbol{\sigma} \quad (1)$$

where $\boldsymbol{\sigma}$ is a vector with the three Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$ (or equivalently written as $(\sigma_x, \sigma_y, \sigma_z)$) as Cartesian components. We use the standard expressions for these 2x2 matrices, as given in the lecture notes. We also introduce the rotated Pauli matrix, defined by $\sigma_{\mathbf{n}} = \mathbf{n} \cdot \boldsymbol{\sigma}$, where \mathbf{n} is any three dimensional unit vector.

a) Show that $\sigma_{\mathbf{n}}$ has eigenvalues ± 1 , and the eigenstate (in matrix form) corresponding to the eigenvalue $+1$ is (up to an arbitrary phase factor)

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} \quad (2)$$

with (θ, ϕ) as the polar angles of the unit vector \mathbf{n} . Also show the relation

$$\Psi_{\mathbf{n}}^\dagger \boldsymbol{\sigma} \Psi_{\mathbf{n}} = \mathbf{n} \quad (3)$$

b) Show, by using operator identities from Problem Set 2, the following relation

$$e^{-\frac{i}{2}\alpha\sigma_z} \sigma_x e^{\frac{i}{2}\alpha\sigma_z} = \cos \alpha \sigma_x + \sin \alpha \sigma_y \quad (4)$$

Explain why this shows that the unitary matrix

$$\hat{U} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = e^{-\frac{i}{\hbar}\alpha\mathbf{n}\cdot\hat{S}} \quad (5)$$

induce a spin rotation of angle α about the axis \mathbf{n} .

c) Demonstrate, by expansion of the exponential function, the following identity

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \cos \frac{\alpha}{2} \mathbb{1} - i \sin \frac{\alpha}{2} \sigma_{\mathbf{n}} \quad (6)$$

with $\mathbb{1}$ as the 2x2 identity matrix.

3.2 Jacobi's Theta Function

Let $g(x)$ denote function in one dimension. We define a related function $f(x)$ by

$$f(x) = \sum_{n=-\infty}^{\infty} g(x + 2\pi n) \quad (7)$$

where we assume the infinite sum to be well defined.

a) Show that f is a periodic function, $f(x + 2\pi) = f(x)$, and therefore can be expressed as a discrete Fourier sum on the interval $0 \leq x < 2\pi$,

$$f(x) = \sum_{l=-\infty}^{\infty} c_l e^{ilx} \quad (8)$$

with

$$c_l = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-ilx} dx \quad (9)$$

b) Assume now $g(x)$ to be specified as a Gaussian function

$$g(x) = k e^{-\lambda x^2} \quad (10)$$

with k as a constant. Find the explicit expression for the Fourier sum (8) in this case. (Use the results for Gaussian integrals from the previous problems.)

We consider now the *elliptic theta function* (or Jacobi theta function) $\theta_3(z|w)$ defined by the sum

$$\theta_3(z|w) = \sum_{l=-\infty}^{\infty} \exp(i\pi w l^2 + 2ilz) \quad (11)$$

c) Show that this sum for θ_3 has the same form as the Fourier sum (8) of the gaussian function (10), with $\phi = 2z$. Find the alternative expression for θ_3 that corresponds to the sum (7).

Show, by use of the above results, that function θ_3 has the following symmetry properties

$$\begin{aligned} \theta_3(z + \pi|w) &= \theta_3(z|w) \\ \theta_3(z + \pi w|w) &= e^{-i\pi w - 2iz} \theta_3(z|w) \\ \theta_3(z|w) &= (-iw)^{-\frac{1}{2}} e^{z^2/i\pi w} \theta_3\left(\frac{z}{w} \middle| -\frac{1}{w}\right) \end{aligned} \quad (12)$$

3.3 Particle on a circle

A particle with mass m moves freely on a circle of radius R .

a) Use the polar angle ϕ as coordinate and find the angular momentum and energy eigenstates $\psi_l(\phi) = \langle \phi | l \rangle$. What is the energy as function of l ? Find an expression for the propagator $\mathcal{G}(\phi, t; 0, 0) = \langle \phi | \hat{\mathcal{U}}(t, 0) | 0 \rangle$ as a sum over angular momenta, by making a direct calculation of the relevant matrix element of the time evolution operator $\hat{\mathcal{U}}(t, 0)$. (The coordinates of the initial position are here chosen as $(\phi_i, t_i) = (0, 0)$.) Show that the propagator can be expressed in terms of the Jacobi theta function $\theta_3(z|w)$.

b) Show that there is an infinity of classical paths, with different *winding numbers* n , that connect the two points $(0, 0)$ and (ϕ, t) , and use the semi-classical expression for the path integral to write the propagator $\mathcal{G}(\phi, t; 0, 0)$ as a sum over winding numbers n . Show that also this sum can be expressed in terms of the Jacobi theta function.

c) Use the results of Problem 3.1 to show that expressions found for the propagator in a) and b) are equivalent.