FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2008

Problem set 7

7.1 Schmidt decomposition

A composite system consists of two two-level systems, $\mathcal{A}$ and $\mathcal{B}$. The Hilbert space of the composite system is spanned by the four tensor product vectors

$$|ij\rangle = |i\rangle_{\mathcal{A}} \otimes |j\rangle_{\mathcal{B}} \;, \quad i = 1,2 \;, j = 1,2 \tag{1}$$

A general pure state can be expanded in this basis as

$$|\psi\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij}|ij\rangle \tag{2}$$

with c_{ij} as the four expansion coefficients.

Assume we make independent transformations on the basis states of the two subsystems so that the new basis vectors are

$$|i\rangle'_{\mathcal{A}} = \sum_j U_{ji}\,|j\rangle_{\mathcal{A}} \;, \quad |i\rangle'_{\mathcal{B}} = \sum_j V_{ji}\,|j\rangle_{\mathcal{B}} \tag{3}$$

with U_{ij} and V_{ij} as matrix elements of two unitary matrices. We denote by d_{ij} the expansion coefficients of $|\psi\rangle$ in the new composite basis $|ij\rangle' = |i\rangle'_{\mathcal{A}} \otimes |j\rangle'_{\mathcal{B}}$.

a) Show that, if we consider c_{ij} and d_{ij} as matrix elements of two 2x2 matrices C and D, then they are related by

$$D = U^\dagger C V^* \tag{4}$$

with U^* as the complex conjugate of the matrix U. By choosing the transformations U and V in a particular form, the matrix D can be brought to diagonal form, $d_{ij} = d_i\,\delta_{ij}$. In the transformed basis the vector $|\psi\rangle$ has only two expansion coeffisients,

$$|\psi\rangle = \sum_{i=1}^{2} d_i |i\rangle'_{\mathcal{A}} \otimes |i\rangle'_{\mathcal{B}} \tag{5}$$

and we refer to this as the Schmidt decomposition of the state vector.

b) Show that the matrix D satisfies the two equations

$$DD^\dagger = U^\dagger C C^\dagger U \;, \quad D^\dagger D = V^T C^\dagger C V^* \tag{6}$$

with $U^T = (U^*)^\dagger$ as the transposed matrix, and explain why we can always find unitary transformations U and V that satisfy these equations.

c) Use the information above to find the Schmidt decomposed form of the following vectors

$$\begin{aligned}
1.\quad |a\rangle &= \frac{1}{\sqrt{2}}(|11\rangle - |22\rangle) \\
2.\quad |b\rangle &= \frac{1}{2}(|11\rangle - |12\rangle - |21\rangle + |22\rangle) \\
3.\quad |c\rangle &= \frac{1}{\sqrt{10}}(2|11\rangle + |12\rangle - |21\rangle - 2|22\rangle)
\end{aligned} \tag{7}$$

7.2 Density operators
Midterm exam 2007, Problem 1

A density operator of a two-level system can be represented by a 2×2 (density) matrix in the form

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})\,, \quad |\mathbf{r}| \leq 1 \tag{8}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\mathbf{r}$ is a vector in three dimensions and $\boldsymbol{\sigma}$ is a vector operator with the Pauli matrices as the Cartesian components. Geometrically the set of all density matrices form of a sphere in three dimensions, with the pure states $|\mathbf{r}| = 1$ as the surface of the sphere (the Bloch sphere), and the mixed states as the interior of the sphere.

a) The density operator can also be expressed in bra-ket formulation as

$$\hat{\rho} = \rho_{11}\,|+\rangle\langle+| \;+\; \rho_{12}\,|+\rangle\langle-| \;+\; \rho_{21}\,|-\rangle\langle+| \;+\; \rho_{22}\,|-\rangle\langle-| \tag{9}$$

where $|\pm\rangle$ is the state of the upper/lower level of the system, that is with $\sigma_z|\pm\rangle = \pm|\pm\rangle$. What are the coefficients $\rho_{ij},\ i,j = 1,2$, expressed in terms of the Cartesian components x, y, z of $\mathbf{r}$?

We consider in the following a composite system with two subsystems $\mathcal{A}$ and $\mathcal{B}$. These are both two-level systems so that the Hilbert space of the full system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is of dimension 4. A density matrix of the composite system can be written as

$$\hat{\rho} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \sum_i a_i\,\sigma_i \otimes \mathbb{1} + \sum_j b_j\,\mathbb{1} \otimes \sigma_j + \sum_{ij} c_{ij}\sigma_i \otimes \sigma_j) \tag{10}$$

with a_i, b_j and c_{ij} as coefficients, and with the first factor in the tensor product corresponding to the $\mathcal{A}$ subsystem and the other to $\mathcal{B}$.

b) Find the reduced density matrices of subsystems $\mathcal{A}$ and $\mathcal{B}$ expressed in terms of the a, b and c coefficients. What condition should the a, b and c coefficients satisfy if the two subsystems should be completely uncorrelated?

We examine the four *Bell states* of the composite system,

$$\begin{aligned} |c,\pm\rangle &= \tfrac{1}{\sqrt{2}}\,(|\,++\rangle \pm |\,--\rangle) \\ |a,\pm\rangle &= \tfrac{1}{\sqrt{2}}(|\,+-\rangle \pm |\,-+\rangle) \end{aligned} \tag{11}$$

where $|ij\rangle = |i\rangle \otimes |j\rangle$, $i,j = \pm$, are tensor product states.

c) Give the expressions for the density operators of the four states, first in the bra-ket form, and then written in the form (10). What are the reduced density matrices of subsystems $\mathcal{A}$ and $\mathcal{B}$ for these four states? Give the entropy of the full system and the two subsystems in the four cases. Why do we call the Bell states *maximally entangled*?

d) We consider linear combinations of the form

$$\hat{\rho} = x\hat{\rho}_1 + (1-x)\hat{\rho}_2 \tag{12}$$

where $\hat{\rho}_1$ and ρ_2 represent two Bell states and x is a real parameter. Show that if we have $0 < x < 1$ the linear combination is a density operator. Why is that not the case if $x < 0$ or $x > 1$?

e) Choose a pair of Bell states and show that halfway between them ($x = 1/2$) the density matrix gets a particularly simple form. Show that it can be written in the form

$$\hat{\rho} = \frac{1}{8}\left[(\mathbb{1} + \mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} + \mathbf{m}\cdot\boldsymbol{\sigma}) + (\mathbb{1} - \mathbf{n}\cdot\boldsymbol{\sigma}) \otimes (\mathbb{1} - \mathbf{m}\cdot\boldsymbol{\sigma})\right] \tag{13}$$

where $\mathbf{n}$ is a unit vector and $\mathbf{m} = \pm\mathbf{n}$. What does this expression show about entanglement between the two subsystems $\mathcal{A}$ and $\mathcal{B}$ for this particular state?

f) The Bell states define a subspace in the space of all 4×4 density matrices. Show that the density matrices in this subspace commute.