FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2008

Problem set 9

9.1 The canonical commutation relations.
The basic commutation relations of the electromagnetic field can be written as

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^{\dagger}_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{1}$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field and a is a polarization index. The photon creation and annihilation operators are related to the field operators by

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(\hat{a}_{\mathbf{k}a} + \hat{a}^{\dagger}_{-\mathbf{k}\bar{a}})\,, \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}(\hat{a}_{\mathbf{k}a} - \hat{a}^{\dagger}_{-\mathbf{k}\bar{a}}) \tag{2}$$

where $\bar{a}$ is related to a by a permutation of the two values of the polarization index. Use the above expressions to show that the creation and annihilation operators satisfy the standard harmonic oscillator commutation relations,

$$\begin{aligned} \left[\hat{a}_{\mathbf{k}a}, \hat{a}^{\dagger}_{\mathbf{k}'a'})\right] &= \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \\ \left[\hat{a}_{\mathbf{k}a}, \hat{a}_{\mathbf{k}'a'})\right] &= \left[\hat{a}^{\dagger}_{\mathbf{k}a}, \hat{a}^{\dagger}_{\mathbf{k}'a'})\right] = 0 \end{aligned} \tag{3}$$

9.2 The electromagnetic field energy and momentum.
The classical expressions for the electromagnetic field energy $\mathcal{E}$ and field momentum $\boldsymbol{\mathcal{P}}$ are

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\int d^3r(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) \\ \boldsymbol{\mathcal{P}} &= \epsilon_0\int d^3r\mathbf{E}\times\mathbf{B} \end{aligned} \tag{4}$$

The same expressions are valid in the quantum description, when the classical fields are replaced by the operator fields, $\mathbf{E}\to\hat{\mathbf{E}}$, $\mathbf{B}\to\hat{\mathbf{B}}$. The energy $\mathcal{E}$ is then replaced by the hamiltonian $\hat{H}$ and the momentum $\boldsymbol{\mathcal{P}}$ by the momentum operator $\hat{\boldsymbol{\mathcal{P}}}$.

Use expressions for the field operators from the lecture notes to show that $\hat{H}$ and $\hat{\boldsymbol{\mathcal{P}}}$ has the following form, when written in terms of the photon number $\hat{N}_{\mathbf{k}a}$ operator

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}a}\hbar\omega_k\hat{N}_{\mathbf{k}a} + E_0\mathbb{1} \\ \hat{\boldsymbol{\mathcal{P}}} &= \sum_{\mathbf{k}a}\hbar\mathbf{k}\hat{N}_{\mathbf{k}a} \end{aligned} \tag{5}$$

where $\hat{N}_{\mathbf{k}a}$ is defined by

$$\hat{N}_{\mathbf{k}a} = \hat{a}^{\dagger}_{\mathbf{k}a}\hat{a}_{\mathbf{k}a} \tag{6}$$

Comment on the interpretation of the constant E_0 and explain the meaning of removing this from the definition of the Hamiltonian, as one usually does.

9.2 The electromagnetic field equation

Show that the field operator $\hat{\mathbf{A}}(\mathbf{r}, t)$, in the Heisenberg picture, satisfies the same field equation as the classical vector potential

$$(\frac{\partial^2}{\partial t^2} - c^2\, \boldsymbol{\nabla}^2)\hat{\mathbf{A}}(\mathbf{r}, t) = 0 \tag{7}$$

9.3 Polarization vectors

Show that the polarization vectors $\boldsymbol{\epsilon}_{\mathbf{k}a}$ satisfy, together with the wave vector $\mathbf{k}$, the following completeness relation,

$$\sum_a (\epsilon_{\mathbf{k}a})_i(\epsilon^*_{\mathbf{k}a})_j + \frac{k_i k_j}{\mathbf{k}^2} = \delta_{ij} \tag{8}$$

9.4 Field commutators

The electric and magnetic fields do not commute, but rather satisfy a commutation relation that can be written as

$$\left[\hat{E}_i(\mathbf{r}), \hat{B}_j(\mathbf{r}')\right] = i\frac{\hbar}{\epsilon_0}\sum_{\mathbf{k}} \epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{r} - \mathbf{r}') \tag{9}$$

with ϵ_{ijk} as the three-dimensional Levi-Civita symbol. Show this by considering the Fourier transform of the equation.

To make precise what is meant by derivatives of delta-functions, one usually consider the corresponding Fourier transformed functions. As a reminder, the Fourier transform of the three-dimensional delta function in a finite volume V is

$$\delta(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{10}$$