FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2009

Problem set 11

11.1 Entanglement

Two persons A and B communicate with the help of quantum entanglement. They share a set of pairs of particles with spins in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \tag{1}$$

where $|++\rangle = |+\rangle \otimes |+\rangle$ is a state where both particles of the pair have *spin up* in the z-direction, and similarly $|--\rangle = |-\rangle \otimes |-\rangle$ is the state where both particles have *spin down* in the z-direction.

a) How do we measure the degree of entanglement in such a two-partite system, and what is the degree of entanglement in the given spin state?

b) Assume A and B perform independent spin operations on their particles in a given pair, each operation described by a unitary operator, $\hat{U}_A$ or $\hat{U}_B$. What happens to the entanglement of the two-particle system under such an operation.

c) Assume A performs an ideal measurement of the spin component in the x- direction, which projects the spin to an eigenstate of the x-component of the spin operator. What happens to the entanglement in this case?

11.2 The canonical commutation relations.

The basic commutation relations of the electromagnetic field can be written as

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{2}$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field and a is a polarization index. The photon creation and annihilation operators are related to the field operators by

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(\hat{a}_{\mathbf{k}a} + \hat{a}^\dagger_{-\mathbf{k}\bar{a}})\,, \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}(\hat{a}_{\mathbf{k}a} - \hat{a}^\dagger_{-\mathbf{k}\bar{a}}) \tag{3}$$

where $\bar{a}$ is related to a by a permutation of the two values of the polarization index. Use the above expressions to show that the creation and annihilation operators satisfy the standard harmonic oscillator commutation relations,

$$\begin{aligned} \left[\hat{a}_{\mathbf{k}a}, \hat{a}^\dagger_{\mathbf{k}'a'})\right] &= \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \\ \left[\hat{a}_{\mathbf{k}a}, \hat{a}_{\mathbf{k}'a'})\right] &= \left[\hat{a}^\dagger_{\mathbf{k}a}, \hat{a}^\dagger_{\mathbf{k}'a'})\right] = 0 \end{aligned} \tag{4}$$

11.3 The electromagnetic field energy and momentum.

The classical expressions for the electromagnetic field energy $\mathcal{E}$ and field momentum $\boldsymbol{\mathcal{P}}$ are

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\int d^3r(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) \\ \boldsymbol{\mathcal{P}} &= \epsilon_0\int d^3r\mathbf{E}\times\mathbf{B} \end{aligned} \tag{5}$$

The same expressions are valid in the quantum description, when the classical fields are replaced by the operator fields, $\mathbf{E} \to \hat{\mathbf{E}}$, $\mathbf{B} \to \hat{\mathbf{B}}$. The energy $\mathcal{E}$ is then replaced by the hamiltonian $\hat{H}$ and the momentum $\boldsymbol{\mathcal{P}}$ by the momentum operator $\hat{\boldsymbol{\mathcal{P}}}$.

Use expressions for the field operators from the lecture notes to show that $\hat{H}$ and $\hat{\boldsymbol{\mathcal{P}}}$ has the following form, when written in terms of the photon number $\hat{N}_{\mathbf{k}a}$ operator

$$\begin{aligned} \hat{H} &= \sum_{\mathbf{k}a} \hbar\omega_k \hat{N}_{\mathbf{k}a} + E_0 \mathbb{1} \\ \hat{\boldsymbol{\mathcal{P}}} &= \sum_{\mathbf{k}a} \hbar\mathbf{k}\hat{N}_{\mathbf{k}a} \end{aligned} \tag{6}$$

where $\hat{N}_{\mathbf{k}a}$ is defined by

$$\hat{N}_{\mathbf{k}a} = \hat{a}^\dagger_{\mathbf{k}a}\hat{a}_{\mathbf{k}a} \tag{7}$$

Comment on the interpretation of the constant E_0 and explain the meaning of removing this from the definition of the Hamiltonian, as one usually does.

11.4 The electromagnetic field equation

Assuming we apply the Coulomb gauge condition, show that the field operator $\hat{\mathbf{A}}(\mathbf{r}, t)$, in the Heisenberg picture, satisfies the same field equation as the classical vector potential

$$(\frac{\partial^2}{\partial t^2} - c^2\, \boldsymbol{\nabla}^2)\hat{\mathbf{A}}(\mathbf{r}, t) = 0 \tag{8}$$

11.4 Polarization vectors

Show that the polarization vectors $\boldsymbol{\epsilon}_{\mathbf{k}a}$ satisfy, together with the wave vector $\mathbf{k}$, the following completeness relation,

$$\sum_a (\epsilon_{\mathbf{k}a})_i(\epsilon^*_{\mathbf{k}a})_j + \frac{k_i k_j}{\mathbf{k}^2} = \delta_{ij} \tag{9}$$

9.5 Field commutators

The electric and magnetic fields do not commute, but rather satisfy a commutation relation that can be written as

$$\left[\hat{E}_i(\mathbf{r}), \hat{B}_j(\mathbf{r}')\right] = i\frac{\hbar}{\epsilon_0}\sum_{\mathbf{k}} \epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{r} - \mathbf{r}') \tag{10}$$

with ϵ_{ijk} as the three-dimensional Levi-Civita symbol. Show this by considering the Fourier transform of the equation.

To make precise what is meant by derivatives of delta-functions, one usually consider the corresponding Fourier transformed functions. As a reminder, the Fourier transform of the threedimensional delta function in a finite volume V is

$$\delta(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{11}$$