FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2009

Problem set 12

12.1 Photon emission.

A particle with mass m and charge e is trapped in a one-dimensional harmonic oscillator potential, oriented in the z-direction. The frequency of the oscillator is ω. At time $t = 0$ the particle is excited to energy level n and it then performs a transition to level $n-1$ by emitting one photon of energy $\hbar\omega$. We write the energy eigenstates of the composite system of charged particle and photons as $|n, n_{\mathbf{k}a}\rangle$, With initially no photon present the state is $|i\rangle = |n, 0\rangle$ while the final state with one photon present is $|f\rangle = |n-1, 1_{\mathbf{k}a}\rangle$. To first order in perturbation theory the angular probability distribution $p(\theta, \phi)$ of the emitted photon is

$$p(\theta, \phi) = \kappa \sum_a |\langle n-1, 1_{\mathbf{k}a}|\hat{H}_{emis}|n, 0\rangle|^2 \tag{1}$$

with (θ, ϕ) as the polar angle of the photon quantum number $\mathbf{k}$ and κ as a proportionality factor. $\hat{H}_{emis}$ is the emission part of the interaction Hamiltonian, which in the dipole approximation is

$$H_{emis} = -\frac{e}{m} \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\epsilon_0\,\omega}}\, \hat{\mathbf{p}} \cdot \boldsymbol{\varepsilon}^*_{\mathbf{k}a}\, \hat{a}^\dagger_{\mathbf{k}a} \tag{2}$$

a) Determine the particle matrix element $\langle n-1|\hat{\mathbf{p}}|n\rangle$.

b) Find the probability distribution $p(\theta, \phi)$.

12.2 Life time of an excited level

We consider atomic trasitions in a single-electron atom. To first order in perturbation theory, and in the dipole approximation, the life time τ_A of an excited level A is determined by

$$1/\tau_A = \sum_B w_{BA} = \sum_B \frac{e^2\omega^3_{AB}}{3\pi\epsilon_0\hbar c^3} \mid \langle B|\mathbf{r}|A\rangle \mid^2 \tag{3}$$

where $\omega_{AB} = (E_A - E_B)/\hbar$ is the frequency of a photon emitted in the transition between atomic levels A and B.

The excited 2p state of the hydrogen atom makes a dipole transition to the ground state 1s. The purpose of this exercise is to evaluate the life time of the 2p state and to find the natural line width of the transition.

When calculating the dipole matrix element, the following information about the wave functions of the atomic states is useful.

In polar coordinates a general energy eigenstate has the form (with the electron spin not included)

$$\psi_{nlm}(r, \theta, \phi) = Y^m_l(\theta, \phi) R_{nl}(r) \tag{4}$$

where $Y^m_l(\theta, \phi)$ is the spherical harmonics with l and m as the orbital angular momentum quantum numbers, and n is the principal (radial) quantum number. For the ground state 1s ($n = 1, \ell = 0$) and the 2p state ($n = 2, \ell = 1$), the radial wave functions are

$$\begin{aligned} R_{10} &= \frac{2}{\sqrt{a_0^3}}\, e^{-\frac{r}{a_0}} \quad (1s) \\ R_{21} &= \frac{1}{\sqrt{24a_0^3}}\, \frac{r}{a_0}\, e^{-\frac{r}{2a_0}} \quad (2p) \end{aligned} \tag{5}$$

$a_0 =$ Bohr radius.

For the angular momentum values $\ell = 0$ and $\ell = 1$ the spherical harmonics are

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}\,, \quad Y_1^{\pm} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}\,, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \tag{6}$$

and for general ℓ and m we have the orthonormality relation

$$\begin{aligned}\int d\phi d\theta \sin\theta\ Y_\ell^m(\theta,\phi)^*\, Y_{\ell'}^{m'}(\theta,\phi) &= \delta_{\ell\ell'}\delta_{mm'} \\ &= \int d\phi d\theta \sin\theta\ Y_\ell^{-m}(\theta,\phi)\, Y_{\ell'}^{m'}(\theta,\phi)\end{aligned} \tag{7}$$

For the matrix element of $\mathbf{r}$, we cite the following result, which appears after integration over the polar angles:

$$\sum_{m'} |\ \langle l'm'n'|\,\mathbf{r}\,|lmn\rangle\ |^2 = \begin{cases} \frac{l+1}{2l+1}\,R_{nl}^{n'l'} & l' = l+1 \\ \frac{l}{2l+1}\,R_{nl}^{n'l'} & l' = l-1 \end{cases} \tag{8}$$

with

$$R_{nl}^{n'l'} = (\int_0^\infty dr r^3 R_{n'l'}(r) R_{nl}(r))^2. \tag{9}$$

a) Show, for two general atomic states $|A\rangle$ and $|B\rangle$, the identity

$$|\ \langle B|\mathbf{r}|A\rangle\ |^2 = \frac{1}{2}\,|\ \langle B|x+iy|A\rangle\ |^2 + \frac{1}{2}\,|\ \langle B|x-iy|A\rangle\ |^2 + |\ \langle B|z|A\rangle\ |^2\,. \tag{10}$$

b) Find expressions for $x \pm iy$ og z in terms of spherical harmonics $Y_1^m(\theta,\phi)$, assume $|A\rangle$ to be the 2p state and $|B\rangle$ to be the 1s state, and verify the expression (8) for the particular case of a transition 2p $\rightarrow$ 1s.

c) Evaluate (9) for this transition and use this to find the life time τ_{2p} of the excited 2p state.

d) Determine the ratio between the natural *spectral line width* $\Gamma_{2p} = \hbar/\tau_{2p}$ of the emission line and the energy difference between the two atomic levels, $E_A - E_B$.

Some useful information is listed:

Energy levels of the hydrogen atom,with m_e as the electron mass,

$$E_n = -\frac{\alpha^2}{2n^2} m_e c^2\,, \quad n = 1,2,3,... \tag{11}$$

Fine structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = 0.00723 \tag{12}$$

Bohr radius,

$$a_0 = \frac{\hbar}{\alpha m_e c} = 0.529\cdot 10^{-10} m \tag{13}$$