FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2009

Problem set 2

2.1 Time dependent transformation
Two unitarily equivalent descriptions of a quantum system are related by a *time dependent* unitary transformation $\hat{U}(t)$, which acts on state vectors as

$$|\psi(t)\rangle \quad \rightarrow \quad |\psi'(t)\rangle = \hat{U}(t)|\psi(t)\rangle \tag{1}$$

and on the observables as

$$\hat{A} \quad \rightarrow \quad \hat{A}'(t) = \hat{U}(t)\,\hat{A}\,\hat{U}(t)^{-1}\,. \tag{2}$$

Show that the Hamiltonian H' which determines the time evolution of the transformed state vector $|\psi'(t)\rangle$ includes an additional term

$$\hat{H} \rightarrow \hat{H}' = \hat{U}(t)\,\hat{H}\,\hat{U}(t)^{-1} + i\hbar\frac{d\hat{U}}{dt}\hat{U}^{-1} \tag{3}$$

2.2 Operator identities
Assume $\hat{A}$ and $\hat{B}$ to be two operators, generally not commuting.
We define the following to composite operators:

$$\begin{aligned}\hat{F}(\lambda) &= e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} \\ \hat{G}(\lambda) &= e^{\lambda\hat{A}}e^{\lambda\hat{B}}\end{aligned} \tag{4}$$

a) Show the following relation

$$\frac{d\hat{F}}{d\lambda} = \left[\hat{A}, \hat{F}\right] \tag{5}$$

and use it to derive the expansion

$$\hat{F}(\lambda) = \hat{B} + \lambda\left[\hat{A}, \hat{B}\right] + \frac{\lambda^2}{2}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] ... \tag{6}$$

b) Show the following relation between $\hat{G}(\lambda)$ and $\hat{F}(\lambda)$,

$$\frac{d\hat{G}}{d\lambda} = (\hat{A} + \hat{F})\hat{G} \tag{7}$$

and use this to demonstrate the following expansion (Campbell-Baker-Hausdorff)

$$\hat{G}(\lambda) = e^{\lambda\hat{A}+\lambda\hat{B}+\frac{\lambda^2}{2}\left[\hat{A},\hat{B}\right]+\,...} \tag{8}$$

by calculating the exponent on the right-hand side to second order in λ.

c) When $\left[\hat{A}, \hat{B}\right]$ commutes with both $\hat{A}$ and $\hat{B}$ the expression (8) is exact without the higher order terms indicated by ... in (8).
Verify this by use of (6) and (7), and by noting that the eigenvalues of $\hat{G}$ satisfy a differential equation that can be integrated.

2.3 Gaussian integral

The following formula gives the integral of a gaussian function

$$I \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \tag{9}$$

This is correct for complex λ provided the real part of λ is positive. Verify this by writing the I^2 as a two-dimensional integral

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\, e^{-\lambda(x^2+y^2)} \tag{10}$$

and by changing to to polar coordinates in the evaluation.

2.4 Gaussian wave function

The quantum state of a free particle with mass m in one dimension is described by the following momentum wave function of Gaussian form,

$$\psi(p) = \sqrt[4]{\frac{\lambda}{\pi}}\, e^{-\frac{\lambda}{2}(p-p_0)^2} \tag{11}$$

where λ is a real, positive parameter that determines the width of the Gaussian. What is the corresponding time dependent wave function $\psi(x, t)$ in the coordinate representation? Show that this function is also a Gaussian, of the form

$$\psi(x) = N' e^{-\frac{\lambda'}{2}(x-x_0)^2} \tag{12}$$

with λ' and x_0 as a time-dependent parameters and N' as a (x-independent) normalization factor. Determine λ' and x_0 and examine how the maximum of the wave packet moves with time and how the width changes. Compare with the classical motion of the particle.