FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2009

Problem set 8

8.1 Spin coherent states
We consider two different sets of basis vectors for a spin-half system. The first one is the standard set of eigenvectors for the Pauli matrix σ_z,

$$\sigma_z|m\rangle = m|m\rangle \ , \quad m = \pm 1 \tag{1}$$

The second set is the set of spin up states for all rotated Pauli matrices, which have been introduced in Sect. 1.3.1 of the lecture notes,

$$\sigma_{\mathbf{n}}|\mathbf{n}\rangle \equiv \mathbf{n}\cdot\boldsymbol{\sigma}|\mathbf{n}\rangle = |\mathbf{n}\rangle \ , \quad \mathbf{n} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k} \tag{2}$$

Clearly this is a much larger set, since it depends on two continuous variables θ and ϕ. (In fact, as discussed in the lecture notes, *any* state of the spin half system can be written in this way.)

The set of states $|\mathbf{n}\rangle$ have several properties similar to the coherent states of a harmonic oscillator, and are therefore referred to as *spin coherent states*. In this problem the objective is to study some of these properties. You may make use of the results of Sect. 1.3.2.

In order to bring the notation closer to that of the coherent states of the harmonic oscillator we represent the unit vector $\mathbf{n}$ by a complex number z in the following way

$$z = e^{i\phi}\cot\frac{\theta}{2} \tag{3}$$

and define $|z\rangle \equiv |\mathbf{n}\rangle$.

a) Show that the transition function between the two sets of states can be written in the form

$$\langle z|m\rangle = \frac{z^{(m+1)/2}}{\sqrt{1+|z|^2}} \tag{4}$$

b) Determine the overlap function

$$|\psi_{z_0}(z)|^2 \equiv |\langle z|z_0\rangle|^2 \tag{5}$$

c) Show that the spin coherent states satisfy a completeness relation of the form

$$\int \frac{d^2z}{2\pi}\frac{4}{(1+|z|^2)^2}|z\rangle\langle z| = \mathbb{1} \tag{6}$$

where d^2z denotes the standard area element in the two-dimensional plane.

8.1 Schmidt decomposition

A composite system consists of two two-level systems, $\mathcal{A}$ and $\mathcal{B}$. The Hilbert space of the composite system is spanned by the four tensor product vectors

$$|ij\rangle = |i\rangle_{\mathcal{A}} \otimes |j\rangle_{\mathcal{B}} \ , \quad i = 1,2 \ , j = 1,2 \tag{7}$$

A general pure state can be expanded in this basis as

$$|\psi\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2} c_{ij}|ij\rangle \tag{8}$$

with c_{ij} as the four expansion coefficients.

Assume we make independent transformations on the basis states of the two subsystems so that the new basis vectors are

$$|i\rangle'_{\mathcal{A}} = \sum_j U_{ji}\,|j\rangle_{\mathcal{A}} \ , \quad |i\rangle'_{\mathcal{B}} = \sum_j V_{ji}\,|j\rangle_{\mathcal{B}} \tag{9}$$

with U_{ij} and V_{ij} as matrix elements of two unitary matrices. We denote by d_{ij} the expansion coefficients of $|\psi\rangle$ in the new composite basis $|ij\rangle' = |i\rangle'_{\mathcal{A}} \otimes |j\rangle'_{\mathcal{B}}$.

a) Show that, if we consider c_{ij} and d_{ij} as matrix elements of two 2x2 matrices C and D, then they are related by

$$D = U^{\dagger}CV^{*} \tag{10}$$

with U^* as the complex conjugate of the matrix U. By choosing the transformations U and V in a particular form, the matrix D can be brought to diagonal form, $d_{ij} = d_i\,\delta_{ij}$. In the transformed basis the vector $|\psi\rangle$ has only two expansion coeffisients,

$$|\psi\rangle = \sum_{i=1}^{2} d_i|i\rangle'_{\mathcal{A}} \otimes |i\rangle'_{\mathcal{B}} \tag{11}$$

and we refer to this as the Schmidt decomposition of the state vector.

b) Show that the matrix D satisfies the two equations

$$DD^{\dagger} = U^{\dagger}CC^{\dagger}U \ , \quad D^{\dagger}D = V^{T}C^{\dagger}CV^{*} \tag{12}$$

with $U^T = (U^*)^{\dagger}$ as the transposed matrix, and explain why we can always find unitary transformations U and V that satisfy these equations.

c) Use the information above to find the Schmidt decomposed form of the following vectors

$$\begin{aligned}
1.\quad |a\rangle &= \frac{1}{\sqrt{2}}(|11\rangle - |22\rangle) \\
2.\quad |b\rangle &= \frac{1}{2}(|11\rangle - |12\rangle - |21\rangle + |22\rangle) \\
3.\quad |c\rangle &= \frac{1}{\sqrt{10}}(2|11\rangle + |12\rangle - |21\rangle - 2|22\rangle)
\end{aligned} \tag{13}$$

8.3 Entangled photons (Midterm Exam 2004)
In this problem correlations between pairs of entangled photons are studied. The interesting degree of freedom is the polarization of each photon. For a single photon this means that the quantum state is a vector in a two-dimensional vector space spanned by the vectors $|H\rangle$ and $|V\rangle$, which correspond to linear polarization in the horizontal and vertical direction, respectively. A general polarization state is a linear combination of these two. As special cases we consider linearly polarized photons in a rotated direction,

$$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{14}$$

and circularly polarized photons with right-handed and left-handed orientation, respectively,

$$|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)\ , \quad |L\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle) \tag{15}$$

The two-photon states, when only polarization is taken into account, are vectors in the tensor product space spanned by the four vectors,

$$\begin{aligned} |HH\rangle &= |H\rangle \otimes |H\rangle\ , \quad |HV\rangle = |H\rangle \otimes |V\rangle\ , \\ |VH\rangle &= |V\rangle \otimes |H\rangle\ , \quad |VV\rangle = |V\rangle \otimes |V\rangle\ , \end{aligned} \tag{16}$$

(Note that even if the photons are bosons there is no symmetry constraint on the two-photon states, since we assume that the two photons can be distinguished by their different direction of propagation.)

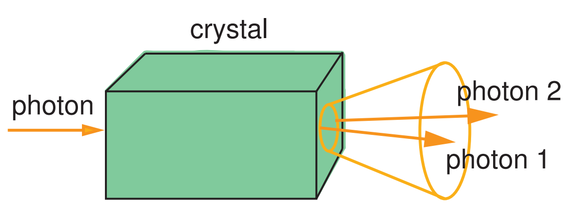

Fig. 2a

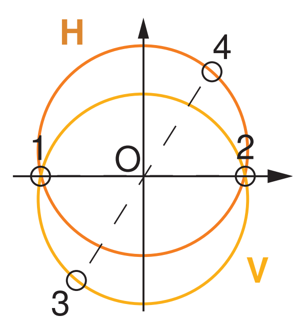

Fig. 2b

As a specific way to produce entangled photon pairs we consider the method of *parametric down conversion*, as described below and sketched in the Figure 2 and 3. As illustrated in Fig. 2a a (weak) beam of photons enter a crystal, where each photon due to the non-linear interaction with the crystal is split into two photons. These appear with equal energy, half the energy of the incoming photon. The transverse momentum of the emerging photons is fixed so that their direction of propagation is limited to a cone, as indicated in the figure. The photons appear with constant probability around the cone. However, due to conservation of total transverse momentum, the two photons in each a pair are correlated so that they always are emitted at opposite sides of the cone.

There is furthermore a polarization effect, since photons with horizontal and vertical polarization (relative to the crystal planes) do not propagate in exactly the same way. As a consequence the cones corresponding to these two polarizations are slightly shifted. This is shown in the head-on view of Fig. 2b, where the cone corresponding to polarization H is slightly lifted relative to the cone corresponding to polarization V.

Two photons in a correlated pair will be located on opposite points of the central point O, like the pair of points 1 and 2 and the pair 3 and 4, and they always appear with orthogonal polarization. As shown by the figure this means that for most directions of the emitted photons the polarization of each photon is uniquely determined by its direction of propagation. For such a pair the two-photon state is a product state of the form $|HV\rangle$. As an illustration, the pair $3, 4$ of directions of the cone, as shown in Fig.2b, will be of this type.

However two directions are different since they lie on both cones. This is illustrated by the points 1 and 2 in Fig. 2b. A photon at one of these positions will be in a superposition of $|H\rangle$ and $|V\rangle$. Due to correlations between the photons a pair located at this position will be described by an entangled two-photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + e^{i\chi}|VH\rangle) \tag{17}$$

where the complex phase χ can be regulated in the experimental set up.

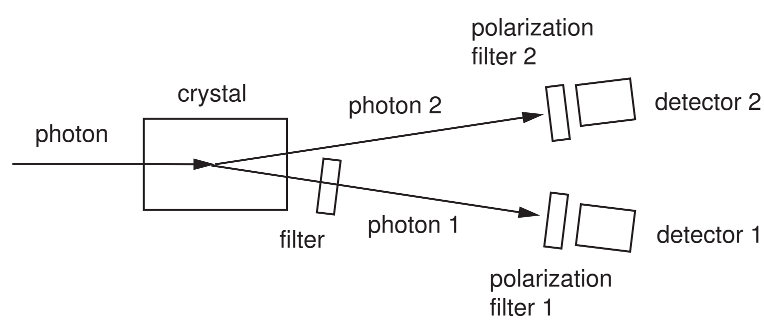

Fig. 3

We assume in the following that a filter close to the crystal will single out photons only in the directions 1 and 2. This is schematically shown in Fig. 3. To analyze correlations between the two photons in each pair, polarization filters are applied to photons in both directions as also shown in the figure. Those that pass the polarization filters are registered in the detectors and the registrations are paired by use of coincidence counters.

The polarization filters may be represented by operators that project on linearly polarized states along a (rotated) direction

$$\hat{P}(\theta) = |\theta\rangle\langle\theta| \tag{18}$$

In the following we examine the expected results of the polarization measurements by calculating the following expectation values

$$\begin{aligned} P_1(\theta_1) &\equiv \left\langle \hat{P}_1(\theta_1) \right\rangle && \text{photon 1} \\ P_2(\theta_1) &\equiv \left\langle \hat{P}_2(\theta_2) \right\rangle && \text{photon 2} \\ P_{12}(\theta_1, \theta_2) &\equiv \left\langle \hat{P}_1(\theta_1)\hat{P}_2(\theta_2) \right\rangle && \text{photon 1 and photon 2} \end{aligned} \tag{19}$$

a) Assume a series of N entangled photon pairs are used in an experiment. In this series n_1 photons are registered in detector 1, n_2 photons are registered in detector 2 and and n_{12} are registered at coincidence in the two detectors. What are the relations between the registered frequences n_1/N, etc. and the the expectation values P_1, P_2 and P_{12}?

b) For the general two-photon state of the form (17) find the density operator of the two-photon pair, and find the corresponding reduced density operators for photon 1 and photon 2.

We consider three different situations where the the incoming photon pairs are in the state (17) with I: $\chi = \pi$, II: $\chi = 0$ and III: $\chi = \pi/2$.

c) Consider an input state of the form I. Determine the detection probability P_1 of photon 1 as a function of the angle θ_1 of polarizer 1. Do the same with P_2 for photon 2. Determine next the probability P_{12} for detecting photons at both analyzers as a function of the angles θ_1 and θ_2. What do the results tell about correlations of the two photons?

d) Consider next a two-photon input state of the form II. Examine the same questions as in c). Are the results obtained rotationally invariant? Compare the cases b) and c).

e) Consider finally the case III. Find also in this case the expectation values P_1, P_2 and P_{12} as fuctions of the angles of the polarizers. Show that in this case there exists a mixed state, which is an *incoherent* mixture of $|HV\rangle$ and $|VH\rangle$, that has identical expectation values.

f) The Bell inequality, which is based on an assumed set of "hidden variables" as a source of the statistical distributions, can be written as a constraint on the function P_{12} in the following way (see Sect. 2.3.2 of the lecture notes),

$$F(\theta_1, \theta_2, \theta_3) \equiv P_{12}(\theta_2, \theta_3) - |P_{12}(\theta_1, \theta_2) - P_{12}(\theta_1, \theta_3)| \geq 0 \tag{20}$$

Examine the Bell inequality in the cases I, II and III for the special choice of angles $\theta_1 = 0$, $\theta_2 = \theta$ and $\theta_3 = 2\theta$ by plotting the function $F(0, \theta, 2\theta)$. Comment on which of the cases that show that the Bell inequality is not satisfied. Is there a relation between the conclusion for the case III and the results in e)?