FYS 4110: Non-relativistic quantum mechanics

Midterm Exam, Fall Semester 2011

The problem set is available from Friday October 14. The set consists of 2 problems written on 4 pages.

Deadline for returning solutions
Friday October 21. *Written/printed* solutions should be returned to Ekspedisjonskontoret in the Physics Building before closing time.

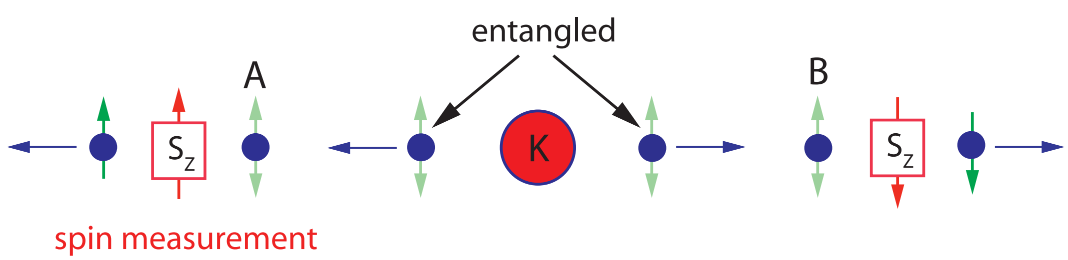
Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room Ø471) or the assistant Marianne Rypestøl (Office: Ø457).

Language
Solutions may be written in Norwegian or English, depending on your preference.

PROBLEMS

1 Entanglement and Bell inequalities

We consider an experimental situation, similar to the one discussed in the lecture notes, where pairs of spin 1/2 particles are initially prepared in a correlated spin state, and then are separated in space while keeping the spin state unchanged. When far apart spin measurements are performed on the particles in each pair, and the results are registered and compared. The situation is illustrated in the figure, where a series of entangled pairs are created in a source K, and where measurements of the z-components of the spin are performed on both particles (A and B). When the spins in the z-directions are strictly anticorrelated, the result *spin up* (*spin down*) for particle A is always followed by the result *spin up* (*spin down*) for particle B.



We consider the situation where three different sets of measurements are performed, with different spin states,

$$\text{I}: \qquad \hat\rho_1 = |\psi_a\rangle\langle\psi_a|, \quad |\psi_a\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

$$\text{II}: \qquad \hat\rho_2 = |\psi_s\rangle\langle\psi_s|, \quad |\psi_s\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

$$\text{III}: \qquad \hat\rho_3 = \frac{1}{2}(\hat\rho_1 + \hat\rho_2) \tag{1}$$

The notation is $| + -\rangle = |+\rangle \otimes |-\rangle$, where $|\pm\rangle$ are spin states of a single particle, with S_z quantized. The first factor in the tensor product refers to particle A and the second one to particle B. Note that all three states are strictly anticorrelated with respect to the z-component of the spin of the two particles. The purpose of the (hypothetical) experiment is to examine correlation functions that are relevant for the Bell inequalities, as already discussed for case I in the lecture notes, to see if the three states show different behavior. This involves performing the spin measurements also for rotated directions of the spin axes.

a) Of the three density operators only $\hat{\rho}_1$ is rotationally invariant. Demonstrate this by calculating the expectation value of $\mathbf{S}^2$ for the three cases, where $\mathbf{S} = (\hbar/2)(\boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{\sigma})$ is the spin vector of the full system, and comment on the results.

b) What are the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$ in the three cases? Determine the von Neuman entropy S of the full system, as well as the entropies S_A and S_B of the subsystems. Check if the classical restriction on the entropies $S \geq \max\{S_A, S_B\}$ is satisfied in any of the cases. In each of the cases examine if the states are entangled or separable, and give, if possible, a numerical measure of the degree of entanglement.

We assume the direction of the two measurement devices can be rotated so they measure spin components of the form

$$S_\theta = \cos\theta S_z + \sin\theta S_x \tag{2}$$

where the angle θ can be chosen independently for A and B. The state $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$ is the *spin up* vector in the rotated direction and the operator $\hat{P}(\theta) = |\theta\rangle\langle\theta|$ projects on the corresponding spin vector.

c) Show that the given expression for $|\theta\rangle$, as claimed above, is the *spin up* state of S_θ. Determine the expectation value $P_A(\theta) = \left\langle \hat{P}(\theta) \right\rangle_A$, for particle A, in the three cases I, II and III. Comment on the result.

d) Determine, for the three cases, the joint probability distribution $P(\theta, \theta') = \left\langle \hat{P}(\theta) \otimes \hat{P}(\theta') \right\rangle$, with the two angles θ and θ' as independent variables.

The Bell inequality, according to the *hidden variable* analysis described in the lecture notes, gives a constraint on the possible classical correlations of the two spins. In the present case the inequality can be written as

$$F(\theta, \theta') \equiv P(0, \theta') - |P(\theta, 0) - P(\theta, \theta')| \geq 0 \tag{3}$$

where one of the angles is set to 0 since we, for the states we consider, will only have strict anticorrelation for spin measurements along the z-axis. (For details see the derivation in the lecture notes.)

e) Make plots of the function $F(\theta, 0.5\,\theta)$ for the three cases I, II and III, with θ varying in the interval $0 < \theta < 2\pi$. Check in all cases whether the inequality (3) is satisfied or broken, and compare the results with what is known from point b) concerning entanglement between the two particles.

In addition to these plots, examine the functions for other choices $\theta' = \lambda\,\theta$ with $\lambda \neq 0.5$ to see if the results are not changed. Alternatively make a 3D plot of the two-variable function $F(\theta, \theta')$ and check whether the conclusion concerning the Bell inequality holds in the full parameter space. State the conclusions, but it is not needed to include the additional plots in the written/ printed solutions.

f) Assume an experimental series is performed, with the two angles fixed. The number of pairs registered with *spin up* (in the chosen direction) for both spins A and B is n_{++}, and the number with *spin down* for both spins is n_{--}. Similarly n_{+-} is the number of pairs registered with *spin up* for A and *spin down* for B, n_{-+} is the number of pairs registered with *spin down* for A and *spin up* for B. The total number of pairs in the series is N.

We refer to the experimental results corresponding to $P_A(\theta)$, $P_B(\theta')$, and $P(\theta,\theta')$ as $P^A_{exp}(\theta)$, $P^B_{exp}(\theta')$, and $P_{exp}(\theta,\theta')$. What are these quantities expressed in terms of the numbers $\{n_{ij}, i,j=\pm\}$ and N?

2 Rabi oscillations in a composite quantum system

An atom interacts with the electromagnetic field within a small reflecting cavity. Only one of the cavity modes of the field has a frequency that matches energy differences between the lowest energy levels of the atom. The interaction can therefore be described by a simplified model, where only two atomic levels are included, denoted $|g\rangle$ (ground state) and $|e\rangle$ (excited state), and only one field mode, with energy levels $|n\rangle$, where n is the photon number of this mode. The model Hamiltonian is

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \equiv \hat{H}_0 + \hat{H}_1 \tag{4}$$

where the $\hat{H}_0$ includes the two first terms, which describe the non-interacting atom and photons, and $\hat{H}_1$ the third term, which describes interactions between the atoms and the photons. $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The model is known as the Jaynes-Cummings model, and it has precisely the form of a two-level system interacting with a harmonic oscillator. The Pauli matrices act between the two atomic levels, with σ_z as the standard diagonal matrix, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. The raising and lowering operators of the harmonic oscillator, $\hat{a}$ and $\hat{a}^\dagger$, have the physical interpretation as photon creation and destruction operators. The model is based on the *rotating wave approximation*, where terms of the form $\hat{a}^\dagger\sigma_+$ and $\hat{a}\sigma_-$ are suppressed since they are unimportant close to resonance, where $\omega_0 \approx \omega$. An unimportant constant energy contribution to $\hat{H}$ has also been subtracted.

a) Show that the interaction Hamiltonian $\hat{H}_1$ couples the unperturbed levels only in pairs that differ by one photon. We define such a pair of states as $|n1\rangle \equiv |g\rangle\otimes|n\rangle = |g,n\rangle$ and $|n2\rangle \equiv |e\rangle\otimes|n-1\rangle = |e,n-1\rangle$ for $n \geq 1$. Show that the Hamiltonian in the subspace spanned by this pair of states can be written as a 2x2 matrix of the form

$$H_n = \frac{1}{2}\hbar\begin{pmatrix}\Delta & i\omega_n \\ -i\omega_n & -\Delta\end{pmatrix} + \epsilon_n\mathbb{1} \tag{5}$$

with $\mathbb{1}$ as the 2×2 identity matrix, and find the expressions for Δ, ω_n and ϵ_n. Assume $|n1\rangle$ to correspond to the upper matrix elements of H_n and $|n2\rangle$ to the lower ones, with the corresponding matrix elements of a state vector referred to as c_{n1} and c_{n2}.

The state $|g,0\rangle = |g\rangle\otimes|0\rangle$ seems not to have any partner. What happens to this state under time evolution?

In the following we assume the resonance condition $\omega = \omega_0$ to be satisfied.

b) Solve the eigenvalue problem for this 2x2 matrix Hamiltonian, and find the two energy eigenvalues $E_n^\pm$ and the corresponding eigenvectors $\phi_n^\pm$ in matrix form. For a general, time dependent state $\psi_n(t)$ find the coefficients $c_{n1}(t)$ and $c_{n2}(t)$ expressed in terms of the coefficients $c_{n1}(0)$ and $c_{n2}(0)$ at the initial time $t=0$.

c) A general state, with all n-components included, can be written as

$$|\psi\rangle = \sum_{n=0}^{\infty}\sum_{i=1}^{2} c_{ni}|ni\rangle \tag{6}$$

What are the corresponding expressions for the matrix elements of the density matrix, $\rho_{ni,n'j}$, Find the expressions also for the reduced matrix elements ρ_{ij} of the atom. What is the physical interpretation of the diagonal terms ρ_{11} and ρ_{22}?

Consider two different initial conditions for a state vector at $t = 0$:

I $|\psi(0)\rangle = |e\rangle \otimes |m-1\rangle$, where m is a specific, but at this point unspecified photon number.

II $|\psi(0)\rangle = |e\rangle \otimes |\alpha\rangle$ with $|\alpha\rangle$ a coherent state, defined by

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} |n\rangle \tag{7}$$

where α is a complex number. Write in both cases the expressions for the reduced density matrix elements $\rho_{ij}(0)$ of the atom.

d) Find the matrix elements of the time dependent, reduced density matrix of the atom $\rho_{ij}(t)$, for both initial conditions I and II.

e) Make plots of $\rho_{11}(t)$ as function of t, for both cases I and II. Make the following choice for the parameters, $\alpha = 4$ and $m = 16$. Use λt as time variable on the horizontal axis. Make a short time plot, $0 < \lambda t < 5$, of both cases in the same diagram. Make also a long time plot for case II, for example with $0 < \lambda t < 100$. Comment on the results and compare with the case discussed in the lecture notes, where the (electro)magnetic field is treated classically.