**FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2011**

## Problem set 10

**1o.1 The canonical commutation relations.**
The basic commutation relations of the electromagnetic field can be written as

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^\dagger_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{1}$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field and $a$ is a polarization index. The photon creation and annihilation operators are related to the field operators by

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(\hat{a}_{\mathbf{k}a} + \hat{a}^\dagger_{-\mathbf{k}\bar{a}})\,, \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}(\hat{a}_{\mathbf{k}a} - \hat{a}^\dagger_{-\mathbf{k}\bar{a}}) \tag{2}$$

where $\bar{a}$ is related to $a$ by a permutation of the two values of the polarization index. Use the above expressions to show that the creation and annihilation operators satisfy the standard harmonic oscillator commutation relations,

$$\begin{aligned}\left[\hat{a}_{\mathbf{k}a}, \hat{a}^\dagger_{\mathbf{k}'a'})\right] &= \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \\ \left[\hat{a}_{\mathbf{k}a}, \hat{a}_{\mathbf{k}'a'})\right] &= \left[\hat{a}^\dagger_{\mathbf{k}a}, \hat{a}^\dagger_{\mathbf{k}'a'})\right] = 0\end{aligned} \tag{3}$$

**10.2 The electromagnetic field energy and momentum.**
The classical expressions for the electromagnetic field energy $\mathcal{E}$ and field momentum $\boldsymbol{\mathcal{P}}$ are

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\int d^3r(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) \\ \boldsymbol{\mathcal{P}} &= \epsilon_0\int d^3r\mathbf{E}\times\mathbf{B}\end{aligned} \tag{4}$$

The same expressions are valid in the quantum description, when the classical fields are replaced by the operator fields, $\mathbf{E} \to \hat{\mathbf{E}}$, $\mathbf{B} \to \hat{\mathbf{B}}$. The energy $\mathcal{E}$ is then replaced by the hamiltonian $\hat{H}$ and the momentum $\boldsymbol{\mathcal{P}}$ by the momentum operator $\hat{\boldsymbol{\mathcal{P}}}$.

Use expressions for the field operators from the lecture notes to show that $\hat{H}$ and $\hat{\boldsymbol{\mathcal{P}}}$ has the following form, when written in terms of the photon number $\hat{N}_{\mathbf{k}a}$ operator

$$\begin{aligned}\hat{H} &= \sum_{\mathbf{k}a}\hbar\omega_k\hat{N}_{\mathbf{k}a} + E_0\mathbb{1} \\ \hat{\boldsymbol{\mathcal{P}}} &= \sum_{\mathbf{k}a}\hbar\mathbf{k}\hat{N}_{\mathbf{k}a}\end{aligned} \tag{5}$$

where $\hat{N}_{\mathbf{k}a}$ is defined by

$$\hat{N}_{\mathbf{k}a} = \hat{a}^\dagger_{\mathbf{k}a}\hat{a}_{\mathbf{k}a} \tag{6}$$

Comment on the interpretation of the constant $E_0$ and explain the meaning of removing this from the definition of the Hamiltonian, as one usually does.

**10.3 Polarization vectors**

Show that the polarization vectors $\boldsymbol{\epsilon}_{\mathbf{k}a}$ satisfy, together with the wave vector $\mathbf{k}$, the following completeness relation,

$$\sum_a (\epsilon_{\mathbf{k}a})_i (\epsilon^*_{\mathbf{k}a})_j + \frac{k_i k_j}{\mathbf{k}^2} = \delta_{ij} \tag{7}$$

**10.4 Squeezed states**

The definition we have used for the coherent states of a harmonic oscillator shows that these states depend on the frequency $\omega$ of the oscillator. It follows from the fact that the raising and lowering operators $\hat{a}^\dagger$ and $\hat{a}$, when expressed in terms of $\hat{x}$ and $\hat{p}$, are frequency dependent, while $\hat{x}$ and $\hat{p}$ are independent of $\omega$.

In this problem we examine this dependence on the frequancy by assuming that a particle in a harmonic oscillator potential, with initial frequency $\omega_a$, is at time $t = 0$ is in a coherent state $|z_0\rangle_a$. At this moment there is a sudden change in the potential to a new frequency $\omega_b$. The quantum state has no time to adjust to this abrupt change, so the state is $|z_0\rangle_a$ also immediately after the frequency has changed.

a) The change of frequency means that the raising and lowering operators are changed. We refer to the operators before the change as $\hat{a}^\dagger$ and $\hat{a}$ and as $\hat{b}^\dagger$ and $\hat{b}$ after the change. Show that we have the following relations between the operators

$$\hat{a} = c\,\hat{b} + s\,\hat{b}^\dagger\,, \quad \hat{a}^\dagger = s\,\hat{b} + c\,\hat{b}^\dagger\,, \tag{8}$$

with the inverse

$$\hat{b} = c\,\hat{a} - s\,\hat{a}^\dagger\,, \quad \hat{a}^\dagger = -s\,\hat{a} + c\,\hat{a}^\dagger\,, \tag{9}$$

and find $c$ and $s$ expressed in terms of the two frequancies $\omega_a$ and $\omega_b$. Explain why we may assume $c$ and $s$ to represent hyperbolic functions of the form $c = \cosh\xi$, $s = \sinh\xi$, for some variable $\xi$.

b) Show that the two sets operators can be related by a unitary transformation

$$\hat{U} = e^{-\frac{\xi}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} = e^{\frac{\xi}{2}(\hat{b}^2 - \hat{b}^{\dagger 2})} \tag{10}$$

so that

$$\hat{a} = \hat{U}\,\hat{b}\,\hat{U}^\dagger\,, \hat{a}^\dagger = \hat{U}\,\hat{b}^\dagger\,\hat{U}^\dagger \tag{11}$$

This is called a Bogoliubov transformation.

c) We now have two sets of coherent states defined by $\hat{a}|z\rangle_a = z|z\rangle_a$ and $\hat{b}|z\rangle_b = z|z\rangle_b$. Show that $|z_0\rangle_a$ is not a coherent state with respect to the new lowering operator $\hat{b}$.

d) We next consider the case $z_0 = 0$, so the initial state is the ground state of the Hamiltonian *before* the change of frequency. Show that this state, when expanded in the energy basis *after* the change, is

$$|0\rangle_a = \sum_{n=0}^{\infty} \alpha_n |n\rangle_b \tag{12}$$

with expansion coefficients

$$\alpha_{2n} = (-\frac{s}{2c})^n \frac{\sqrt{(2n)!}}{n!}\alpha_0 \tag{13}$$

and

$$\alpha_0^{-2} = \sum_{n=0}^{\infty} (\frac{s}{2c})^{2n} \frac{(2n)!}{(n!)^2} \tag{14}$$

e) In the original coherent state representation the state $|0\rangle_a$ is represented as the wave function $\psi_0^a(z) = \langle z_a|0\rangle_a$ and in the later coherent state representation as $\psi_0^b(z) = \langle z_b|0\rangle_a$. Use the above expansion to find an expression for $\psi_0^b(z) = \langle z_b|0\rangle_a$ as a sum over $n$. Choose the numerical value $s = \frac{1}{2}$, with the corresponding value for $c$, and make a numerical 3D or contour plot of the absolute value $|\psi_0^b(z)|^2$, with the real and imaginary components of $z$ as variables, to get a picture of the form of the function. Compare with a similar plot of $|\psi_0^a(z)|^2 = e^{-|z-z_0|^2}$ and suggest an explanation for the name *squeezed state* used for $\psi_0^b(z)$.

f) When time evolves for $t > 0$ the state $\psi_0^b(z)$ will rotate in the complex $z$-plane. Give a simple demonstration of this and comment on what is the frequency of rotation.

**10.5 Coupled harmonic oscillators (Exam 2009)**

Two harmonic oscillators, called $\mathcal{A}$ og $\mathcal{B}$, are treated as a composite quantum system. The Hamiltonian has the form

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \mathbb{1}) + \hbar\lambda(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \tag{15}$$

with $(\hat{a}, \hat{a}^\dagger)$ as lowering and raising operators for $\mathcal{A}$ and $(\hat{b}, \hat{b}^\dagger)$ as corresponding operators for $\mathcal{B}$. $\omega$ og $\lambda$ are two real constants.

a) Show that the Hamiltonian can be written in the diagonal form,

$$\hat{H} = \hbar\omega_c\, \hat{c}^\dagger\hat{c} + \hbar\omega_d\, \hat{d}^\dagger\hat{d} + \hbar\omega\mathbb{1} \tag{16}$$

with $c$ and $d$ as linear combinations of $a$ and $b$,

$$c = \mu\, a + \nu\, b\,, \quad d = -\nu\, a + \mu\, b \tag{17}$$

and with $\mu$ and $\nu$ as real constants that satisfy $\mu^2 + \nu^2 = 1$. (Corresponding expressions are valid for the hermitian conjugate operators $\hat{c}^\dagger$ og $\hat{d}^\dagger$.) Determine the new parameters $\omega_c$, $\omega_d$, $\mu$ og $\nu$, expressed in terms of $\omega$ og $\lambda$. Check that the new operators $\hat{c}$ og $\hat{d}$ satisfy the same commutation relations as $\hat{a}$ og $\hat{b}$, so that $\left[\hat{c}, \hat{c}^\dagger\right] = \left[\hat{d}, \hat{d}^\dagger\right] = \mathbb{1}$ og $\left[\hat{c}, \hat{d}^\dagger\right] = 0$.

b) Assume the state $|\psi(0)\rangle$ of the composite system at $t = 0$ is a coherent state of both variables, so that

$$\hat{c}\,|\psi(0)\rangle = z_{c\,0}|\psi(0)\rangle\,, \quad \hat{d}\,|\psi(0)\rangle = z_{d\,0}|\psi(0)\rangle \tag{18}$$

The state will also at a later time be a coherent state for $\hat{c}$ og $\hat{d}$, with eigenvalues

$$z_c(t) = e^{-i\omega_c t} z_{c\,0}\,, \quad z_d(t) = e^{-i\omega_d t} z_{d\,0} \tag{19}$$

Show this for $z_c(t)$. (The expression for $z_d(t)$ can be found in the same way, and is therefore not needed to be shown.)

c) Show that the state $|\psi(t)\rangle$ is a coherent state also for the original harmonic oscillator operators $\hat{a}$ og $\hat{b}$, and determine the eigenvalues $z_a(t)$ og $z_b(t)$ expressed in terms of the initial values $z_{a\,0}$ og $z_{b\,0}$.