FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2011

Problem set 12

12.1 Electric dipole transition in hydrogen (Exam 2008)

We consider the transition in hydrogen from the excited 2p level to the ground state 1s, where a single photon is emitted. The initial atomic state (A) we assume to have $m = 0$ for the z-component of the orbital angular momentum, so that the quantum numbers of this state are $(n, l, m) = (2, 1, 0)$, with n as the principle quantum number and l as the orbital angular momentum quantum number. Similarly the ground state (B) has quantum numbers $(n, l, m) = (1, 0, 0)$. When expressed in polar coordinates the wave functions of the two states (with intrinsic spin of the electron not included) are given by

$$\psi_A(r, \phi, \theta) = \frac{1}{\sqrt{32\pi a_0^3}} \cos\theta \, \frac{r}{a_0} \, e^{-\frac{r}{2a_0}}$$
$$\psi_B(r, \phi, \theta) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} \tag{1}$$

where a_0 is the Bohr radius.

We remind you about the form of the interaction matrix element in the dipole approximation,

$$\langle B, 1_{\mathbf{k}a} | \hat{H}_{emis} | A, 0\rangle = ie\sqrt{\frac{\hbar\omega}{2V\epsilon_0}}\, \boldsymbol{\epsilon}_{\mathbf{k}a} \cdot \mathbf{r}_{BA} \tag{2}$$

where e is the electron charge, $\mathbf{k}$ is the wave vector of the photon, a is the polarization quantum number, ω is the photon frequency and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector. V is a normalization volume for the electromagnetic wave functions, ϵ_0 is the permittivity of vacuum and $\mathbf{r}_{BA}$ is the matrix element of the electron position operator between the initial and final atomic states.

a) Explain why the x- and y-components of $\mathbf{r}_{BA}$ vanish while the z-component has the form $z_{BA} = \nu a_0$, with ν as a numerical factor. Determine the value of ν. (A useful integration formula is $\int_0^\infty dx\, x^n\, e^{-x} = n!$.)

b) To first order in perturbation theory the interaction matrix element (2) determines the direction of the emitted photon, in the form of a probability distribution $p(\phi, \theta)$, where (ϕ, θ) are the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi, \theta)$ from the above expressions.

c) The life time of the 2p state is $\tau_{2p} = 1.6 \cdot 10^{-9} s$ while the excited 2s state (with angular momentum $l = 0$) has a much longer life time, $\tau_{2s} = 0.12 s$. Du you have a (qualitative) explanation for the large difference?

12.2 Spin flip radiation (Exam 2010)

We examine here the transition between two spin states for an electron in an external magnetic field directed along the z-axis, $\mathbf{B} = B\mathbf{e}_z$. (Note: $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ here denote the unit vectors along the x, y and z axes, while $\mathbf{k}$ is the wave vector of the emitted photon.) The Hamiltonian is $\hat{H} = \hat{H}_0 + \hat{H}_1$, with $\hat{H}_0$, corresponding to the magnetic dipole energy in the external magnetic field, and $\hat{H}_1$ giving the coupling between the spin and and the radiation field. We have

$$\hat{H}_0 = \frac{1}{2}\omega_B \sigma_z\,, \qquad \omega_B = -\frac{eB}{m} \tag{3}$$

with e as the electron charge and m as the electron mass. The frequency ω_B is regarded as positive.

The matrix element of the interaction Hamiltonian $\hat{H}_1$ corresponding to the emission of a single photon is in the dipole approximation

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle = i\frac{e\hbar}{2m}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\,(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA} \tag{4}$$

where $|A\rangle$ is the excited spin state (spin up) and $|B\rangle$ is the ground state (spin down). Furthermore $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector and $\omega = c\,k$ is the circular frequency of the emitted photon, V is the normalization volume of the electromagnetic field and $\boldsymbol{\sigma}_{AB}$ is the matrix element of the Pauli matrix $\boldsymbol{\sigma} = \sigma_x\,\mathbf{e}_x + \sigma_y\,\mathbf{e}_y + \sigma_z\,\mathbf{e}_z$, taken between the two spin states. As a reminder, the standard form of the Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

a) To first order in the perturbation the angular dependence of the matrix element $|\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle|^2$ will determine the probability distribution for the direction of the emitted photon, $p(\phi,\theta)$, with (ϕ,θ) as the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi,\theta)$ from the above expression. We remind you about the summation rule for the polarization vectors, $\sum_a |\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{b}\cdot\frac{\mathbf{k}}{k}|^2$ with $\mathbf{b}$ as an arbirary vector. The normalization of the probability distribution is $\int d\phi \int d\theta \sin\theta\, p(\phi,\theta) = 1$.

b) The absolute square of the matrix element also determines, for a given $\mathbf{k}$, the probability distribution for the direction of polarization of the photon. Assume that a photon detector registers the photon emitted in the direction of the x-axis ($\mathbf{k} = k\mathbf{e}_x$) with polarization along the vector $\boldsymbol{\epsilon}(\alpha) = \cos\alpha\,\mathbf{e}_y + \sin\alpha\,\mathbf{e}_x$. What is the probability distribution $p(\alpha)$ of detecting the photon, expressed as a function of the angle α? (Assume also here that the distribution is normalized to 1, which means that it gives the probabilities for different polarization states, provided the photon is emitted along the x-aksen.)

c) To a good approximation the occupation probability of the excited spin state will decay exponentially with time

$$P_A(t) = e^{-t/\tau_A} \tag{6}$$

where the life time τ_A to first order in the interaction is determined by the (constant) transition rate

$$w_{BA} = \frac{V}{(2\pi\hbar)^2}\int d^3k \sum_a |\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle|^2 \delta(\omega - \omega_B) \tag{7}$$

Use this to find an expression for the life time τ_A.