FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2011

Problem set 2

2.1 Operator identities

Assume $\hat{A}$ and $\hat{B}$ to be two operators, generally not commuting.
We define the following to composite operators:

$$\begin{aligned} \hat{F}(\lambda) &= e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} \\ \hat{G}(\lambda) &= e^{\lambda \hat{A}} e^{\lambda \hat{B}} \end{aligned} \tag{1}$$

a) Show the following relation

$$\frac{d\hat{F}}{d\lambda} = \left[\hat{A}, \hat{F}\right] \tag{2}$$

and use it to derive the expansion

$$\hat{F}(\lambda) = \hat{B} + \lambda \left[\hat{A}, \hat{B}\right] + \frac{\lambda^2}{2} \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] ... \tag{3}$$

b) Show the following relation between $\hat{G}(\lambda)$ and $\hat{F}(\lambda)$,

$$\frac{d\hat{G}}{d\lambda} = (\hat{A} + \hat{F})\hat{G} \tag{4}$$

and use this to demonstrate the following expansion (Campbell-Baker-Hausdorff)

$$\hat{G}(\lambda) = e^{\lambda \hat{A} + \lambda \hat{B} + \frac{\lambda^2}{2}\left[\hat{A}, \hat{B}\right] + \, ...} \tag{5}$$

by calculating the exponent on the right-hand side to second order in λ.

c) When $\left[\hat{A}, \hat{B}\right]$ commutes with both $\hat{A}$ and $\hat{B}$ the expression (5) is exact without the higher order terms indicated by ... in (5).
Verify this by use of (3) and (4), and by noting that the eigenvalues of $\hat{G}$ satisfy a differential equation that can be integrated.

2.2 Gaussian integral

The following formula gives the integral of a gaussian function

$$I \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \tag{6}$$

This is correct for complex λ provided the real part of λ is positive. Verify this by writing the I^2 as a two-dimensional integral

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, e^{-\lambda(x^2+y^2)} \tag{7}$$

and by changing to to polar coordinates in the evaluation.

2.3 Gaussian wave function

The quantum state of a free particle with mass m in one dimension is, at time $t = 0$, described by the following momentum wave function of Gaussian form,

$$\psi(p) = \sqrt[4]{\frac{\lambda}{\pi}}\, e^{-\frac{\lambda}{2}(p-p_0)^2} \tag{8}$$

where λ is a real, positive parameter that determines the width of the Gaussian. What is the corresponding time dependent wave function $\psi(x, t)$ in the coordinate representation? Show that this function is also a Gaussian, of the form

$$\psi(x) = N' e^{-\frac{\lambda'}{2}(x-x_0)^2} \tag{9}$$

with λ' and x_0 as a time-dependent parameters and N' as a (x-independent) normalization factor. Determine λ' and x_0 and examine how the maximum of the wave packet moves with time and how the width changes. Compare with the classical motion of the particle.

2.4 Time dependent Hamiltonian

A particle moves in one dimension under the influence of a time dependet oscillating force. The Hamiltonian is

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_0^2(\hat{x} - a\sin\omega t)^2 \tag{10}$$

with m as the particle mass, and ω_0, ω and a as constants.

a) Find the form of Heisenberg's equation of motion for the observables $\hat{x}$ and $\hat{p}$, and show that the motion is the same as for a classical harmonic oscillator which is subject to a periodic external force.

b) The Hamiltonian (10) determines the time evolution of the state vectors in the Schrödinger picture. We now change to another representation of the system, by use of a time dependent unitary transformation $\hat{U}(t)$ that acts on state vectors and observables in the Schrödinger picture. The form of the unitary operator is

$$U(t) = e^{\frac{i}{\hbar}(\alpha(t)\hat{p} - \beta(t)\hat{x})} \tag{11}$$

with $\alpha(t)$ and $\beta(t)$ as time dependent variables. Assume that the pair of variables (α, β) satisfy the same equations of motion as the Heisenberg equation of motion satisfied by the pair $(\hat{x}, \hat{p})$. Show that the Hamiltonian $\hat{H}'$, after the transformation has the form

$$\hat{H}' = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_0^2\hat{x}^2 + f(t)\mathbb{1} \tag{12}$$

and determine the function f expressed in terms of α and β.

c) Show that the time time evolution operator, in the transformed reference frame, has the form

$$\hat{\mathcal{U}}(t, t_0)' = \exp\{-\frac{i}{\hbar}[(\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega_0^2\hat{x}^2)(t - t_0) + (g(t) - g(t_0))\mathbb{1}]\} \tag{13}$$

where $g(t) = \int dt f(t)$.

d) Assume that initially (for $t = 0$) the wave function is identical to the ground state wave function $\psi_0(x)$ of the harmonic oscillator Hamiltonian (10), with $a = 0$. Use the results above to determine the time evolution of the wave function (in the Schrödinger picture). Compare the motion of the wave function to the motion of the driven oscillator, with $a \neq 0$, which is discussed in a). (For simplicity set $\alpha(0) = \beta(0) = 0$.)