FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2011

Problem set 3

3.1 Dirac's delta function

The basic relation defining the delta functions is the following

$$f(x) = \int_{-\infty}^{\infty} dx'\, \delta(x - x')\, f(x') \tag{1}$$

with $f(x)$ is any chosen function. Clearly this is not a function in the usual sense, and in particular it has the property that $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$. Nevertheless it is possible (with some care) to treat is a function and as we know from the wave function description of quantum physics it is in many cases a very useful concept.

In the following the formula for Fourier transformation in one dimension is useful

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx} \tag{2}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \tag{3}$$

a) Show that the delta function has the following Fourier expansion

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \tag{4}$$

b) Assume $g(x)$ is a differentiable function with zeros at a set of points x_i,

$$g(x_i) = 0 \quad \text{for} \quad i = 1, 2, ..., N \tag{5}$$

Assume also that the derivative does not vanish at these points, $g'(x_i) \neq 0$. Show by use of the defininition (1) that we have the following relation

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \tag{6}$$

3.2 Position and momentum eigen states

The position and momentum eigen states are given by the relations

$$\hat{x}|x\rangle = x|x\rangle \qquad \langle x|x'\rangle = \delta(x - x') \tag{7}$$

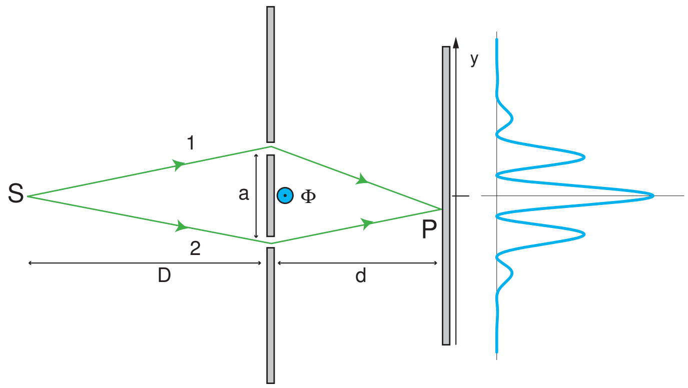
$$\hat{p}|p\rangle = p|p\rangle \qquad \langle p|p'\rangle = \delta(p - p') \tag{8}$$

Furthermore, in the x-representation the momentum operator is given by $\hat{p} = -i\hbar \frac{d}{dx}$. Use these relations together with the Fourier expansion of the delta function to show that the scalar product of a momentum and a position state is give by

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} xp} \tag{9}$$

3.3 The Aharonov-Bohm effect

We consider a double slit experiment as sketched in the figure. Electrons are emitted from a source S and can pass trough one of the two slits of a first screen before being registered on a second screen. When a large number of electrons are registered they are found to form an interference patterns with minima and maxima on the screen.



Behind the middle part of the first screen a solenoid is placed which carries a magnetic flux Φ. The direction of the solenoid is parallel to the direction of the two slits, so that the paths through the upper slit pass on one side of the solenoid and the paths through the lower slit pass on the other side of the solenoid. We consider the magnetic field to be completely screened from the region where the electrons move, so that at no point along the trajectories of the electron there is a magnetic force acting on the particles. Nevertheless, quantum theory predicts that the strength of the magnetic flux will influence the interference pattern so that the maxima and minima are shifted up or down when the flux is changed. This is called the Aharonov-Bohm effect.

We consider in the following the distance d between the screens and the distance D between the source and the first screen to be much larger than the distance a between the two slits, and also to be much larger than the distance y from the central point of the second screen to any point P where an electron is registered.

As a reminder the classical Lagrangian of an electron moving in a magnetic field is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 + e\mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} \tag{10}$$

with $\mathbf{A}$ as the vector potential, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. As follows from Stokes' theorem the magnetic flux is given as the line integral

$$\Phi = \oint_C \mathbf{A} \cdot d\mathbf{r} \tag{11}$$

where C is any given closed loop that encircles once the solenoid.

We consider the situation where a single electron is emitted at time $t = 0$ from the source and is registered at a later time t at a point P of the screen. The probability distribution over the screen for where the electron is registered can be written as

$$p(y) = \lambda |\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0)|^2 \tag{12}$$

with y as the vertical coordinate of P, λ as a proportionality factor and $\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0)$ as the propagator from the initial point $(\mathbf{r}_S, 0)$ to the final point $(\mathbf{r}_P, t)$.

We consider in the following the the semi-classical approximation to the propagator, which we write as

$$\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0) = N \sum_{i=1}^{2} e^{\frac{i}{\hbar} S_i} \tag{13}$$

where S_i is the action integral for classical free-particle motion either through the upper slit ($i = 1$) or through the lower slit $i = 2$, and N is a (y-dependent) normalization factor which is assumed to be independent of the path. Since the classical motion is not affected by the magnetic field, both λ and N are independent of the magnetic flux.

a) Show that the probability $p(y)$ depends on the *difference* between the action integrals of the two paths.

b) Show that the difference between the two action integrals can be written as a function of the magnetic flux Φ.

c) Show that the probability $p(y)$ depends periodically on the magnetic flux Φ. What is the flux period? Describe qualitatively how the interference pattern changes with variations in Φ.

3.4 Spin operators and Pauli matrices

A spin half operator $\hat{\mathbf{S}}$ is defined in the standard way as

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\boldsymbol{\sigma} \tag{14}$$

where $\boldsymbol{\sigma}$ is a vector with the three Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$ (or equivalently written as $(\sigma_x, \sigma_y, \sigma_z)$) as Cartesian components. We use the standard expressions for these 2x2 matrices, as given in the lecture notes. We also introduce the rotated Pauli matrix, defined by $\sigma_{\mathbf{n}} = \mathbf{n} \cdot \boldsymbol{\sigma}$, where $\mathbf{n}$ is any three dimensional unit vector.

a) Show that $\sigma_{\mathbf{n}}$ has eigenvalues ± 1, and the eigenstate (in matrix form) corresponding to the eigenvalue $+1$ is (up to an arbitrary phase factor)

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{15}$$

with (θ, ϕ) as the polar angles of the unit vector $\mathbf{n}$. Also show the relation

$$\Psi_{\mathbf{n}}^{\dagger}\boldsymbol{\sigma}\Psi_{\mathbf{n}} = \mathbf{n} \tag{16}$$

b) Show, by using operator identities from Problem Set 2, the following relation

$$e^{-\frac{i}{2}\alpha\sigma_z}\,\sigma_x\,e^{\frac{i}{2}\alpha\sigma_z} = \cos\alpha\,\sigma_x + \sin\alpha\,\sigma_y \tag{17}$$

Explain why this shows that the unitary matrix

$$\hat{U} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = e^{-\frac{i}{\hbar}\alpha\mathbf{n}\cdot\hat{\mathbf{S}}} \tag{18}$$

induce a spin rotation of angle α about the axis $\mathbf{n}$.

c) Demonstrate, by expansion of the exponential function, the following identity

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \cos\frac{\alpha}{2}\mathbb{1} - i\sin\frac{\alpha}{2}\sigma_{\mathbf{n}} \tag{19}$$

with $\mathbb{1}$ as the 2x2 identity matrix.