FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2011

Problem set 4

4.1 Poisson summation

Consider an unspecified function $g(x)$ in one dimension. We define the related function $f(x)$ by

$$f(x) = \sum_{n=-\infty}^{\infty} g(x + 2\pi n) \tag{1}$$

where we assume the infinite sum to be well defined.

a) Show that f is a periodic function, $f(x + 2\pi) = f(x)$, and therefore can be expressed as a discrete Fourier sum on the interval $0 \leq x < 2\pi$,

$$f(x) = \sum_{l=-\infty}^{\infty} c_l \, e^{ilx} \tag{2}$$

with coefficients

$$c_l = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x) e^{-ilx} dx \tag{3}$$

b) Show from this the identity

$$\sum_{n=-\infty}^{\infty} g(n) = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{-2\pi ilx} dx \tag{4}$$

This is expression is known as Poisson summation, and when the integral can be performed it gives in many cases a useful resummation of the original sum.

c) Assume $g(x)$ to be specified as a Gaussian function

$$g(x) = k e^{-\lambda x^2} \tag{5}$$

with k and λ as a constants. Find the explicit expression for the Fourier sum (2) in this case. (Use the results for Gaussian integrals from one of the earlier problem sets, and consider λ to have a positive real part in order for the integral to converge.)

4.2 Jacobi's Theta Function

As a preparation for Problem 4.3 we consider here the symmetries of a special function called the *Jacobi Theta Function* θ_3. The function $\theta_3(z|w)$ is defined by the sum

$$\theta_3(z|w) = \sum_{l=-\infty}^{\infty} \exp(i\pi w l^2 + 2ilz) \tag{6}$$

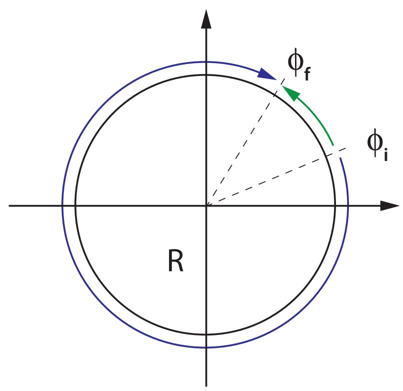
c) Show that this sum has the same form as the Fourier sum (2) of the gaussian function (5), with $x = 2z$, and show that the alternative expression for θ_3 that corresponds to the sum (1) is

$$\theta_3(z|w) = \sqrt{\frac{i}{w}} \sum_{n=-\infty}^{\infty} \exp[-\frac{i}{\pi w}(z + \pi n)^2] \tag{7}$$

b) Show, by use of the above results, that function θ_3 has the following symmetry properties

$$\begin{aligned}\theta_3(z+\pi|w) &= \theta_3(z|w) \\ \theta_3(z+\pi w|w) &= e^{-i\pi w-2iz}\theta_3(z|w) \\ \theta_3(z|w) &= \sqrt{\frac{i}{w}}\, e^{-iz^2/(\pi w)}\, \theta_3(\frac{z}{w}|-\frac{1}{w})\end{aligned} \tag{8}$$

4.3 Particle on a circle
A particle with mass m moves freely on a circle of radius R.



a) Use the polar angle ϕ as coordinate and find the expresssion for the angular momentum eigenstates, $\psi_l(\phi) = \langle\phi|l\rangle$, with l as the dimensionless angular momentum quantum number (which take integer values). These states are also energy eigenstates. What is the energy E_l, expressed in terms of of l?

b) Find an expression for the propagator $\mathcal{G}(\phi,t;0,0) = \langle\phi|\hat{\mathcal{U}}(t,0)|0\rangle$ as a sum over angular momenta, by making a direct calculation of the relevant matrix element of the time evolution operator $\hat{\mathcal{U}}(t,0)$. (The coordinates of the initial position are here chosen as $(\phi_i,t_i) = (0,0)$.) Show that the propagator can be expressed in terms of the Jacobi theta function $\theta_3(z|w)$.

c) Explain why there is an infinity of classical paths, with different *winding numbers* n, that connect the two points $(0,0)$ and (ϕ,t), and use the semi-classical expression for the path integral to write the propagator $\mathcal{G}(\phi,t;0,0)$ as a sum over winding numbers n. Show that also this sum can be expressed in terms of the Jacobi theta function.

d) Use the results of Problem 4.2 to show that expressions found for the propagator in a) and b) are equivalent. This demonstrates that the semi-classical expression for the propagator of a free particle on a circle, in the same way as for a free particle on a line, is identical to the exact expression for the propagator.

4.4 Harmonic oscillator states
The Hamiltonian of a one-dimensional harmonic oscillator is given by

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega^2\hat{x}^2) \tag{9}$$

a) Introduce the lowering and raising operators

$$\hat{a} = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega\hat{x} + i\hat{p})\,, \quad \hat{a}^\dagger = \sqrt{\frac{1}{2m\omega\hbar}}(m\omega\hat{x} - i\hat{p}) \tag{10}$$

and show that the Hamiltonian can be written as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})) \tag{11}$$

b) The energy eigenvectors $|n\rangle$ are defined by the equation

$$\hat{H}|n\rangle = E_n|n\rangle \tag{12}$$

Show that $E_n = \hbar\omega(n + \frac{1}{2})$, and that the raising and lowering operators satisfy the relations

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \tag{13}$$

b) In the energy representation (or n representation), the energy eigenvectors $|n\rangle$ are used as a complete, orthonormal basis. A general observable $\hat{A}$ in this basis can be expressed as an infinite matrix with matrix elements

$$A_{mn} = \langle m|\hat{A}|n\rangle \tag{14}$$

Find the expressions for the (m, n) matrix elements for the following operators
$\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2$ and $\hat{x}\hat{p} + \hat{p}\hat{x}$. Write the operators in matrix form with the 4x4 submatrix corresponding to $n = 0, 1, 2, 3$ written out explicitely.

c) All energy eigenstates can be generated from the ground state by use of the relations,

$$\hat{a}|0\rangle = 0\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{15}$$

Write these equations in the coordinate representation (x-representation), where the energy eigenstates are represented by wave functions $\psi_n(x) = \langle x|n\rangle$. Use the equations to show that the eigenstates in this representation have the form

$$\psi_n(x) = P_n(x)e^{-\lambda x^2} \tag{16}$$

with $P_n(x)$ as a polynomial of order n in x. Find λ and $P_n(x)$ for the three lowest states, $n = 0, 1, 2$.