UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 4110/ 9110 Non-relativistic quantum mechanics
Day of exam: Wednesday, December 7, 2011
Exam hours: 4 hours, beginning at 14:30
This examination paper consists of 2 problems on 3 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Dressed photon states

A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are admixtures of the photon and atomic states. These are sometimes referred to as a "dressed" photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{1}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{2}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the 2×2 identity matrix. Express the parameters Δ, $\cos\phi$, $\sin\phi$, and ϵ in terms of ω_0, ω and λ.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of ϕ, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.

PROBLEM 2
A radiation problem

We consider a one-dimensional problem where a two-level system (A) interacts with a scalar radiation field (B). The notation we use is essentially the same as in Problem 1. The Hamiltonian of the system we consider is

$$\hat{H} = \frac{1}{2}\hbar\omega_A\,\sigma_z + \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k + \kappa\sum_k \sqrt{\frac{\hbar}{2L\omega_k}}(\hat{a}_k\sigma_+ + \hat{a}_k^\dagger\sigma_-) = \hat{H}_0 + \hat{H}_{int} \tag{3}$$

The first term is the two-level Hamiltonian, with energy splitting $\hbar\omega_A$, the second one is the free field contribution, with $k = 2\pi n/L$ (n - integer) as the wave number of the photon. L is a (large) normalization length. The third term is the interaction term $\hat{H}_{int}$, with κ as an interaction parameter. The frequency parameter is $\omega_k = ck$.

a) A general state of the two-level system is characterized by a vector $\mathbf{r}$, with $r \leq 1$, and with the corresponding density matrix as

$$\rho_A = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{4}$$

Consider first that the interaction term $\hat{H}_{int}$ is turned off, $\kappa = 0$, so that the time evolution operator of the two-level system is $\hat{\mathcal{U}}(t) = \exp(-\frac{i}{2}\omega_A\,t\,\sigma_z)$. Use this to determine the the density matrix $\rho_A(t)$ at time t, assuming that $\rho_A(0)$ is identical to the density matrix in (4), and show that the time evolution of $\mathbf{r}$ is a precession around the z-axis with angular velocity ω_A.

b) Assume next that $\kappa \neq 0$ and that initially the two-level system is in the excited "spin up state", while the scalar field is in the vacuum state. Thus, the initial state is $|+,0\rangle = |+\rangle\otimes|0\rangle$. It decays to the "spin down state" by emission of a field quantum. The final state we then write as $|-,1_k\rangle = |-\rangle\otimes|1_k\rangle$.

The occupation probability of the excited state $|+\rangle$ decays exponentially, $P_+(t) = \exp(-\gamma t)$, with a decay rate γ that to first order in the interaction, and in the limit $L\to\infty$, is given by

$$\gamma = \frac{L}{(2\pi\hbar)^2}\int dk\,|\langle -,1_k|\hat{H}_{int}|+,0\rangle|^2\,\delta(\omega_k - \omega_A) \tag{5}$$

Determine the decay rate γ, expressed in terms of the parameters of the problem.

As discussed in the lectures an approximate way to handle the decay is to introduce an imaginary contribution to the energy of the decaying state. Assuming a more general initial state, of the form

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle) \otimes |0\rangle = \alpha|+,0\rangle + \beta|-,0\rangle \tag{6}$$

with α and β as unspecified coefficients, with $|\alpha|^2 + |\beta|^2 = 1$, we make the corresponding *ansatz* for the time evolved state

$$|\psi(t)\rangle = (e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle) \otimes |0\rangle + \sum_k c_k(t)|-,1_k\rangle \tag{7}$$

with $c_k(t)$ as decay parameters, which satify $c_k(0) = 0$.

c) Check what normalization of the state vector (7) means for the decay parameters, and determine the reduced density matrix matrix $\rho_A(t)$ of the two-level system.

d) Assume the same initial conditions as in b), $z(0) = 1$, $x(0) = y(0) = 0$ ($\alpha = 1$, $\beta = 0$). Determine the density matrix $\rho_A(t)$ and the corresponding time dependent vector $\mathbf{r}(t)$. Is the time evolution consistent with the expected exponential decay of the excited state of the two-level system? Give a brief description of the evolution of the entanglement between the two level system and the radiation field during the decay.

e) Choose another initial condition $x(0) = 1$, $y(0) = z(0) = 0$ ($\alpha = \beta = 1/\sqrt{2}$), and find also in this case the time evolution of the reduced density matrix and the components of the vector $\mathbf{r}(t)$. Sketch the time evolution of $\mathbf{r}(t)$ and compare qualitatively the motion with that in a) and d). Find $r(t)^2$ expressed as a function of γt, and sketch also this function. What does it show about the time evolution of the entanglement between the two subsystems A and B?

Assume in this paragraph $\gamma << \omega_A$.

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PROBLEM 1
Two spin-half systems

A quantum system is composed of two interacting spin-half systems. The Hamiltonian has the form

$$\hat{H} = \frac{1}{2}\hbar\omega_1\,\sigma_z \otimes \mathbb{1} + \frac{1}{2}\hbar\omega_2\,\mathbb{1} \otimes \sigma_z + \frac{1}{2}\hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

where σ_z og $\sigma_\pm$ are Pauli matrices, with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hbar\omega_1$ and $\hbar\omega_2$ giving the splitting between the two energy levels of each of the spins, and with λ as a coupling parameter. The two factors of the tensor product refer to each of the two spin systems. We define the frequency difference as $\Delta = \omega_1 - \omega_2$ and introduce the following parametrization, $\Delta = \mu\cos\phi$ and $\lambda = \mu\sin\phi$. We further use $|\pm\rangle$ as notation for the eigenstates of σ_z. In the following we use the tensor products of these states as basis for the Hilbert space of the composite system.

a) Show that only the product states $|+-\rangle = |+\rangle \otimes |-\rangle$ and $|-+\rangle = |+\rangle \otimes |-\rangle$ are mixed by the λ term in the Hamiltonian, and show that the mixing coefficients only depend on the angle ϕ, which we will assume to lie in the interval $0 \leq \phi \leq \pi/2$. Give the expression for the Hamiltonian as a 2x2 matrix, when restricted to the subspace spanned by $|+-\rangle$ and $|-+\rangle$.

b) Find the corresponding two energy eigenvalues, and find the eigenstates expressed as functions of ϕ.

c) We now assume $\Delta = 0$. At time $t = 0$ the system is in the state $|+-\rangle$. Determine the time evolution of the state vector and the corresponding reduced density matrices for the two subsystems. Show that the entanglement entropy has a periodic behavior. What are the maximum and minimum values and what is the period of the oscillations.

PROBLEM 2
Atom-photon interaction in a cavity

An atom is trapped inside a small reflecting cavity. The energy difference between the ground state and the first excited state is $\Delta E = E_e - E_g \equiv \hbar\omega$, with ω matching the frequency of one of the electromagnetic cavity modes. This gives a strong coupling between the atomic states and this cavity mode, while the couplings to the other cavity modes are weak and can be neglected.

The composite system, atom plus cavity mode, is described by the following effective Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) - i\gamma\hbar a^\dagger a \tag{2}$$

where the Pauli matrices act between the two atomic levels, with σ_z being diagonal in the energy basis, and $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ being matrices that raise or lower the atomic energy. $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and destruction operators. λ is an interaction parameter and γ is a decay parameter. The decay is due to the process where the photon escapes through the cavity walls. Both λ and γ are real-valued parameters, and we assume $\gamma << \lambda$ and $\gamma << \omega$.

We characterize the relevant states of the composite system as $|g,0\rangle$, $|g,1\rangle$ and $|e,0\rangle$, where g refers to the atomic ground state, e to the excited state, and 0 and 1 refers to the absence or presence of a photon in the cavity mode.

a) Show that in the two-dimensional subspace spanned by the vectors $|g,1\rangle$ and $|e,0\rangle$ the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} -i\gamma & \lambda \\ \lambda & i\gamma \end{pmatrix} \tag{3}$$

where $|g,1\rangle$ corresponds to the upper row of the matrix and $|e,0\rangle$ to the lower one, and $\mathbb{1}$ is the identity matrix.

b) Assume that initially the system is in the state $|\psi(0)\rangle = |e,0\rangle$. Show that the time evolution of the state vector can be written as

$$|\psi(t)\rangle = e^{-\frac{i}{2}\omega t - \frac{1}{2}\gamma t}\left((\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle + ib\sin(\Omega t)|g,1\rangle\right) \tag{4}$$

and determine the constants Ω, a and b.

c) Denote the corresponding density operator as $\hat{\rho}(t)$. The norm of this operator is not conserved, but if we add a contribution

$$\hat{\rho}_{tot}(t) = \hat{\rho}(t) + f(t)|g,0\rangle\langle g,0| \tag{5}$$

then the norm is conserved, with value 1, for a particular function $f(t)$. Determine this function, and comment on in what sense the addition of the last term in (5) is reasonable, when considering the physical process described by the Hamiltonian (3). Give a short qualitative description of the process described by (5).

PROBLEM 3
Distributed information

A secret message is distributed to a party of three, denoted A, B, and C, in the form of an entangled three-spin state, coded into three spin-half particles. As the receiving party knows in advance, the quantum state is one out of a selection of three,

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}(|+--\rangle + \eta^n|-+-\rangle + (\eta^*)^n|--+\rangle)\,, \quad \eta = e^{2\pi i/3} \tag{6}$$

where $n = 0, 1, 2$. The message is identified by the value of n, which means by which of the three quantum states that is distributed.

We use the notation $|+--\rangle = |+\rangle \otimes |-\rangle \otimes |-\rangle$ etc., where the single spin states $|\pm\rangle$ are orthogonal states in a basis referred to as *basis I*. The three spinning particles are distributed to A, B and C, one particle to each of them, with the the first state in the tensor product corresponding to the spin sent to A, the second one to B and the third one to C. We assume the three-spin state is preserved under this distribution.

Each person in the receiving party can make (spin) measurements on the spinning particle he/she receives. The three can also communicate over a classical channel, which means that they can correlate their measurements and also compare the results of the measurements. They have, however no quantum channel available for communication. This means that all the observables that are available for measurements by the receiving party are of product form.

a) Determine the reduced density operator of A, and explain why, for any measurement he/she performs on his particle, no information can be extracted about which of the three spin states $|\psi_n\rangle$ is distributed. Also show that if A, B and C all make their spin measurements in *basis I*, even if they communicate their measured results, these cannot make any distinction between the three values of n.

Next, consider the situation where A and B are not able to communicate with C. They decide to perform measurements on the two spins they have received, and to make a probabilistic evaluation for the different values of n, based on the measured results. In order to do so they decide both to make their spin measurements in a rotated basis, which we refer to as *basis II*. The vectors in this basis are

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{7}$$

The possible outcomes of the measurements they list with numbers $k = 1, 2, 3, 4$, with the correspondence

$$k = 1:\ (0\,,0)\,, \quad k = 2:\ (0\,,1)\,, \quad k = 3:\ (1\,,0)\,, \quad k = 4:\ (1\,,1) \tag{8}$$

We refer to the corresponding states as $|\phi_k\rangle$, with $|\phi_1\rangle = |00\rangle = |0\rangle \otimes |0\rangle$, etc.

Before they do the measurements they evaluate for each three-spin state $|\psi_n\rangle$ the probabilities for the different measurement results (labeled by k). These probabilities are referred to as $p(k|n)$.

b) Find the reduced density operator $\hat{\rho}_n^{AB}$ and determine the probabilities $p(k|n)$ for different values of k and n. It is sufficient, due to repetitions of results, to consider $n = 0, 1$ and $k = 1, 2$.

Do you, in particular, see a reason why the probabilities are the same for $n = 1$ and $n = 2$, for all k?

c) Assume now that A and B perform their measurements, with the result labeled by k. The probability for the state to be $|\psi_n\rangle$, under the condition that the measured result is k, we denote by $\bar{p}(n|k)$. Under the assumption that all spin states $|\psi_n\rangle$ are equally probable until the result of the measurement is known, statistics theory gives us the following relation

$$\bar{p}(n|k) = \frac{p(k|n)}{p(k)} \tag{9}$$

with $p(k)$ as a normalization factor. Determine $p(k)$ and the probability $\bar{p}(n|k)$ for each n in the case $k = 1 : \ (0\ , 0)$. What is most probably the message that has been distributed?

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Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 4110/9110 Ikke-relativistisk kvantemekanikk
Eksamensdag: Mandag 9. desember, 2013
Tid for eksamen: 4 timer, fra kl. 14:30
Oppgavesettet består av 3 oppgaver på 3 sider
Tillatte hjelpemidler: Godkjent kalkulator
Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

OPPGAVE 1
Tidsutvikling i et to-nivåsystem

Hamiltonoperatoren for et isolert to-nivåsystem (betegnet A) har formen $\hat{H}_0 = (1/2)\hbar\omega\,\sigma_z$, med σ_z som den diagonale Paulimatrisen. Vi betegner den normerte grunntilstandsvektoren som $|g\rangle$ og den eksiterte tilstanden som $|e\rangle$. Systemet er i realiteten koblet til et strålingsfelt (betegnet S), og den eksiterte tilstanden vil derfor henfalle til grunntilstanden under utsendelse av et strålingskvant. Vi lar $\hat{\rho}$ betegne den reduserte tetthetsoperatoren til delsystem A. Med god tilnærmelse kan tidsutviklingen av denne beskrives av den såkalte Lindbladligningen, her på formen

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[H_0, \hat{\rho}\right] - \frac{1}{2}\gamma\left[\hat{\alpha}^\dagger\hat{\alpha}\hat{\rho} + \hat{\rho}\hat{\alpha}^\dagger\hat{\alpha} - 2\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger\right] \tag{1}$$

med γ som henfallsraten for overgangen $|e\rangle \to |g\rangle$, $\hat{\alpha} = |g\rangle\langle e|$ og $\hat{\alpha}^\dagger = |e\rangle\langle g|$.

På matriseform, i basis $\{|e\rangle, |g\rangle\}$, skriver vi tetthetsoperatoren $\hat{\rho}$ som

$$\hat{\rho} = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{2}$$

med p_e som sannsynligheten for å finne systemet i tilstand $|e\rangle$ og p_g som sannsynligheten for å finne det i tilstand $|g\rangle$.

a) Anta først at to-nivåsystemet ved tiden $t = 0$ er i tilstanden $\hat{\rho} = |e\rangle\langle e|$. Vis ved bruk av ligning (1) at sannsynligheten p_e avtar eksponensielt, med γ som henfallsrate, mens total sannsynlighet $p_e + p_g$ er bevart.

b) Anta så en annen initialtilstand hvor to-nivåsystemet ved $t = 0$ er i den rene tilstanden $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Bestem den tidsavhengige tetthetsmatrisen $\hat{\rho}(t)$ med denne initialtilstanden.

c) Tetthetsoperatoren for system A kan alternativt uttrykkes ved Paulimatrisene, som $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$. Bestem funksjonen $r^2(t)$ i de to tilfellene ovenfor og vis at den i begge tilfeller har minimum for $t = (1/\gamma)\ln 2$. Hva blir minimalverdien til r i de to tilfellene? Gi en kommentar om hva dette sier om sammenfiltringen mellom systemene A og S. (Vi forutsetter at det fulle systemet A+S hele tiden er i en ren tilstand.)

OPPGAVE 2
Tre partikler i en sammenfiltret tilstand

Tre partikler med halvtallig spinn, som vi referer til som $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, befinner seg i en sammensatt spinntilstand

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle \; + \; |ddd\rangle) \tag{3}$$

hvor $|uuu\rangle = |u\rangle_{\mathcal{A}} \otimes |u\rangle_{\mathcal{B}} \otimes |u\rangle_{\mathcal{C}}$, er tensorproduktet av *spinn-opp (u)* langs z-aksen for alle tre partiklene, mens $|ddd\rangle = |d\rangle_{\mathcal{A}} \otimes |d\rangle_{\mathcal{B}} \otimes |d\rangle_{\mathcal{C}}$ er tensorprodukttilstanden svarende til *spinn-ned (d)* for alle tre partiklene. Vi antar at posisjonskoordinatene er helt frakoblet spinnkoordinatene og at spinnet derfor kan studeres separat. Det er ingen vekselvirkning mellom partiklene, og tilstanden (3) er derfor uendret så lenge det ikke måles på noen av spinnene.

Vi studerer i det følgende spinnsystemet som todelt, svarende til en oppsplitting $\mathcal{ABC} = \mathcal{A} + \mathcal{BC}$, slik at spinn $\mathcal{A}$ definerer det ene undersystemet og de to andre spinnene, $\mathcal{B}$ og $\mathcal{C}$, definerer det andre undersystemet.

a) Bestem de reduserte tetthetsoperatorene $\hat{\rho}_{\mathcal{A}}$ og $\hat{\rho}_{\mathcal{BC}}$ for de to delsystemene, og sammenfiltringsentropien til det sammensatte systemet. Hva menes med at de to delsystemene i denne tilstanden er *maksimalt* sammenfiltret? Delsystemet $\mathcal{BC}$ kan videre tenkes sammensatt av undersystemene $\mathcal{B}$ og $\mathcal{C}$. Hva sier tetthetsoperatoren $\hat{\rho}_{\mathcal{BC}}$ om sammenfiltring mellom disse to.

b) Ved et gitt tidspunkt blir en spinnmåling utført på partikkel $\mathcal{A}$ som bestemmer spinnkomponenten langs x-aksen som *spinn-opp* langs denne aksen. Denne informasjonen medfører at tetthetsoperatoren til systemet $\mathcal{BC}$ blir endret. Hva blir den nye reduserte tetthetsoperator $\hat{\rho}'_{\mathcal{BC}}$? Har måling på spinnet til partikkel $\mathcal{A}$ forandret sammenfiltringen mellom $\mathcal{B}$ og $\mathcal{C}$?

Vi minner om følgende: Med $|f\rangle$ som spinn-opp langs x-aksen og $|b\rangle$ som spinn-ned langs samme akse har vi relasjonene

$$|u\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle)\,, \quad |d\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) \tag{4}$$

c) Anta at tre-spinnsystemet igjen befinner seg i tilstanden (3). Denne gangen måles spinnet til $\mathcal{A}$ langs en akse i xz-planet, som er rotert med vinkelen θ i forhold til z-aksen. Anta også at i dette tilfellet er måleresultatet *spinn-opp*. Finn hvordan måleresultatet nå påvirker tetthetsoperatoren for systemet $\mathcal{BC}$, og bestem sammenfiltringsentropien for sammensetningen $\mathcal{B} + \mathcal{C}$, som funksjon av vinkelen θ.

For de roterte spinntilstandene gjelder

$$\begin{aligned} |\theta, +\rangle &= \cos(\theta/2)|u\rangle + \sin(\theta/2)|d\rangle \qquad \text{(spinn opp)} \\ |\theta, -\rangle &= -\sin(\theta/2)|u\rangle + \cos(\theta/2)|d\rangle \quad \text{(spinn ned)} \end{aligned} \tag{5}$$

der $\theta = 0$ svarer til kvantisert spinn langs z-aksen og $\theta = \pi/2$ til kvantisert spinn langs x-aksen.

OPPGAVE 3
Spinnflipp-stråling

Vi studerer i denne oppgaven overgang mellom to spinntilstander for et elektron i et ytre magnetfelt som er rettet langs z-aksen, $\mathbf{B} = B\mathbf{e}_z$. (Merk: vi benytter her $\mathbf{e}_x$, $\mathbf{e}_y$ og $\mathbf{e}_z$ som enhetsvektorer langs x-, y- og z-aksen.) Hamiltonoperatoren skrives som $\hat{H} = \hat{H}_0 + \hat{H}_1$, hvor $\hat{H}_0$ svarer til den magnetiske dipolenergien i det ytre magnetfeltet, mens $\hat{H}_1$ beskriver koblingen mellom spinnet og strålingsfeltet. Vi har

$$\hat{H}_0 = \frac{1}{2}\omega_B \sigma_z\,, \qquad \omega_B = -\frac{eB}{m} \tag{6}$$

med e som elektronladningen og m som elektronmassen. Frekvensen ω_B regnes som positiv.

Matriseelementet til spinnvekselvirkningen $\hat{H}_1$, ved emisjon av et foton, er i dipoltilnærmelsen

$$\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle = i\frac{e\hbar}{2m}\sqrt{\frac{\hbar}{2\omega V\epsilon_0}}\,(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}a})\cdot\boldsymbol{\sigma}_{BA} \tag{7}$$

hvor $|A\rangle$ er den eksiterte spinntilstanden (spinn-opp) og $|B\rangle$ er grunntilstanden (spinn-ned). Videre er $\mathbf{k}$ bølgetallsvektoren, $\boldsymbol{\epsilon}_{\mathbf{k}a}$ en polarisasjonsvektor og $\omega = c\,k$ er vinkelfrekvensen til det emitterte fotonet. V er et normeringsvolum for den elektromagnetiske strålingen og $\boldsymbol{\sigma}_{AB}$ er matriseelementet til Paulimatrisen $\boldsymbol{\sigma} = \sigma_x\,\mathbf{e}_x + \sigma_y\,\mathbf{e}_y + \sigma_z\,\mathbf{e}_z$ mellom de to spinntilstandene.

a) Til første orden i perturbasjonsteori vil vinkelavhengigheten til det kvadrerte matriseelementet (summert over polarisasjonsindeksen) $\sum_a |\langle B, 1_{\mathbf{k}a}|\hat{H}_1|A, 0\rangle|^2$, bestemme sannsynlighetsfordelingen for retningen til fotonet, $p(\phi, \theta)$, hvor (ϕ, θ) er polarvinklene til bølgevektoren $\mathbf{k}$. Bestem $p(\phi, \theta)$ fra uttrykket ovenfor. Vi minner om at ved summasjon over polarisasjonsretningene har vi $\sum\limits_a |\boldsymbol{\epsilon}_{\mathbf{k}a}\cdot\mathbf{b}|^2 = |\mathbf{b}|^2 - |\mathbf{b}\cdot\frac{\mathbf{k}}{k}|^2$ for en vilkårlig vektor $\mathbf{b}$. Normeringen av sannsynlighetsfordelingen er $\int d\phi \int d\theta \sin\theta\, p(\phi, \theta) = 1$.

b) Det kvadrerte matriseelementet (uten sum over a) bestemmer også, for gitt $\mathbf{k}$, sannsynlighetsfordelingen over polarisasjonsretningen til fotonet. Anta at en fotondetektor registrerer fotoner utsendt langs x-aksen ($\mathbf{k} = k\mathbf{e}_x$), med polarisasjonsretning $\boldsymbol{\epsilon}(\alpha) = \cos\alpha\,\mathbf{e}_y + \sin\alpha\,\mathbf{e}_z$. Hva er sannsynligheten $p(\alpha)$ for å detektere det emitterte fotonet? Anta her at sannsynligetsfordelingen er normert slik at summen over to ortogonale retninger er, $p(\alpha) + p(\alpha + \pi/2) = 1$. Hva sier resultatet om polarisasjonen til det emitterte fotonet?

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Vi minner om standardformen på Paulimatrisene,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 4110 / 9110 Ikke-relativistisk kvantemekanikk
Eksamensdag: Mandag 8. desember, 2014
Tid for eksamen: 4 timer, fra kl. 14:30
Oppgavesettet består av x oppgaver på y sider
Tillatte hjelpemidler: Godkjent kalkulator
Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

1 Entanglement in a two-spin system

We consider a composite quantum system consisting of two spin-half systems, A and B. The relevant states are restricted to the two-dimensional subspace spanned by the two (orthogonal) Bell states

$$|1\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)\,, \quad |2\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \tag{1}$$

where we use the notation $|+-\rangle = |+\rangle \otimes |-\rangle$, with $|\pm\rangle$ refering to the two eigenstates of σ_z.

Consider first (Case I) a linear superposition of the two state vectors, of the form

$$|\psi(x)\rangle = \cos x\,|1\rangle + \sin x\,|2\rangle\,, \quad 0 \le x \le \frac{\pi}{2} \tag{2}$$

The corresponding density operator we denote by $\hat\rho_I(x) = |\psi(x)\rangle\langle\psi(x)|$.

a) Determine the reduced density operators $\hat\rho_{IA}(x)$ and $\hat\rho_{IB}(x)$ of the two spins and the corresponding entropies $S_{IA}(x)$ and $S_{IB}(x)$. Characterize the entanglement of the two spins for the special values $x = 0$, $\pi/4$, and $\pi/2$.

Consider next (Case II) the following linear combination of the density operators of the two Bell states,

$$\hat\rho_{II}(x) = \cos^2 x\,|1\rangle\langle 1| + \sin^2 x\,|2\rangle\langle 2|\,, \quad 0 \le x \le \frac{\pi}{2} \tag{3}$$

b) What is the von Neuman entropy of this state? Find the reduced density operators $\hat\rho_{IIA}(x)$ and $\hat\rho_{IIB}(x)$, and the corresponding entropies $S_{IIA}(x)$, and $S_{IIB}(x)$. Characterize also here the states of the full system for $x = 0$, $\pi/4$, and $\pi/2$.

For a composite quantum system in pure quantum state, the degree of entanglement is expressed by the von Neumann entropy of one of its subsystems. When the system is in a mixed state we do not have a general, universally accepted, measure for the degree of entanglement. However, for a classical, statistical system we have the following inequality for the entropy of the full systems and its subsystem,

$$\Delta \equiv S - \max\{S_A, S_B\} \geq 0 \tag{4}$$

The breaking of this inequality in quantum system therefore indicates that the two subsystems are entangled.

c) Show that in the two cases I and II the functions $\Delta_I(x)$ and $\Delta_{II}(x)$ are negative for all x, except for one value of x.

OPPGAVE 2
Radiation damping

A charged particle is oscillating in a one-dimensional harmonic oscillator potential. It emits electric dipole radiation, with the rate for transition between an initial state i and a final state f given by the radiation formula

$$w_{fi} = \frac{4\alpha}{3c^2}\omega_{fi}^3 |x_{fi}|^2 \tag{5}$$

where α is the fine structure constant, $\hbar\omega_{fi}$ is the energy radiated in the transition, and c is the speed of light. x is the position coordinate of the particle, which is related to the raising and lowering operators of the harmonic oscillator by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \tag{6}$$

with m as the mass of the particle.

a) Show that the non-vanishing transition rates are of the form

$$w_{n-1,n} = \gamma n \tag{7}$$

with $n = 0, 1, 2, ...$ as referring to the energy levels of the harmonic oscillator, and γ as a constant decay parameter. Detemine γ.

The effect of the radiation on the state of the oscillating particle is described by the Lindblad equation in the following way

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H_0, \hat{\rho}] - \frac{1}{2}\gamma\left[\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right] \tag{8}$$

with $\hat{\rho}$ as the density operator of the particle and H_0 as the harmonic oscillator Hamiltonian, without decay.

b) In the following we focus on the diagonal terms of the density matrix, $p_n = \rho_{nn} = \langle n|\hat{\rho}|n\rangle$, which define the occupation probabilities of the energy eigenstates. Show that they satisfy the equation

$$\frac{dp_n}{dt} = -\gamma(np_n - (n+1)p_{n+1}) \tag{9}$$

Explain why this is consistent with the expression (7) for the transition rate $w_{n-1,n}$.

c) Show that Eq. (9) implies that the expectation value of the excitation energy

$$E = \langle H_0 \rangle - \frac{1}{2}\hbar\omega \tag{10}$$

decays exponentially with time.

OPPGAVE 3
A state in thermal equilibrium

A quantum state in thermal equilibrium is described by the density operator

$$\hat{\rho}(\beta) = N(\beta) e^{-\beta \hat{H}} = N(\beta) \sum_n e^{-\beta E_n} |n\rangle\langle n| \tag{11}$$

with $\hat{H}$ as the Hamiltonian, E_n as the corresponding energy eigenvalues, and $N(\beta)$ as a normalization factor. The parameter β is related to the temperature T by $\beta = 1/(k_B T)$, with k_B as Boltzmann's constant.

a) Show that the expectation value for the energy can be expressed in terms of $N(\beta)$ as

$$E(\beta) = \frac{d}{d\beta} \ln N(\beta) \tag{12}$$

and find a similar expression for the von Neumann entropy $S(\beta) = \mathrm{Tr}[\hat{\rho}(\beta) \ln \hat{\rho}(\beta)]$. (Use here the natural logarithm in the definition of S.)

b) For a two-level system, with Hamiltonian $\hat{H} = (\epsilon/2)\, \sigma_z$, determine the functions $N(\beta)$, $E(\beta)$ and $S(\beta)$, and make a sketch of the expectation value of the energy E as function of the temperature T.

c) Find the density operator expressed in the form $\hat{\rho} = (1/2)(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})$. Determine $\mathbf{r}$ as a function of β and relate this to the results in b).