EXAM in FYS 4110/9110 Modern Quantum Mechanics 2015
Solutions

PROBLEM 1

a) Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega(\sigma_z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z) + \hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

Action on the basis states

$$\begin{aligned} \hat{H}|++\rangle &= \hat{H}|--\rangle = 0 \\ \hat{H}|+-\rangle &= \hbar\omega|+-\rangle + \hbar\lambda|-+\rangle \\ \hat{H}|-+\rangle &= -\hbar\omega|-+\rangle + \hbar\lambda|+-\rangle \end{aligned} \tag{2}$$

Matrix form of $\hat{H}$

$$H = \hbar\begin{pmatrix} \omega & \lambda \\ \lambda & -\omega \end{pmatrix} = \hbar a\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{3}$$

b) Eigenvalue equation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \epsilon\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

Secular equation

$$\epsilon^2 - \cos^2\theta - \sin^2\theta = 0 \quad \Rightarrow \quad \epsilon = \pm 1 \equiv \epsilon_\pm \tag{5}$$

Energy eigenvalues

$$E_\pm = \pm\hbar a = \pm\hbar\sqrt{\omega^2 + \lambda^2} \tag{6}$$

Eigenvectors

$$\begin{aligned} \cos\theta\alpha_\pm + \sin\theta\beta_\pm &= \pm\alpha_\pm \\ \Rightarrow \quad \alpha_+/\beta_+ = (1+\cos\theta)/\sin\theta &= \cot\frac{\theta}{2} \\ \alpha_-/\beta_- = (-1+\cos\theta)/\sin\theta &= -\tan\frac{\theta}{2} \end{aligned} \tag{7}$$

$$\begin{aligned} \Rightarrow \quad |\psi_+\rangle &= \cos\frac{\theta}{2}|+-\rangle + \sin\frac{\theta}{2}|-+\rangle \\ |\psi_-\rangle &= \sin\frac{\theta}{2}|+-\rangle - \cos\frac{\theta}{2}|-+\rangle \end{aligned} \tag{8}$$

The states $|++\rangle$ and $|--\rangle$ are energy eigenstates with eigenvalues $E = 0$.

c) Product states

$$\hat{\rho}_1 = |++\rangle\langle++|\,, \quad \hat{\rho}_2 = |--\rangle\langle--| \tag{9}$$

have no entanglement. Reduced density operators

$$\hat{\rho}_1^A = \hat{\rho}_1^B = |+\rangle\langle+|\,, \quad \hat{\rho}_2^A = \hat{\rho}_2^B = |-\rangle\langle-| \tag{10}$$

Non-product states

$$\begin{aligned} \hat{\rho}_\pm = |\psi_\pm\rangle\langle\psi_\pm| &= \cos^2\frac{\theta}{2}|+-\rangle\langle+-| + \sin^2\frac{\theta}{2}|-+\rangle\langle+-| \\ &\pm \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}(|+-\rangle\langle-+| + |-+\rangle\langle+-|) \end{aligned} \tag{11}$$

Reduced density operators

$$\begin{aligned} \hat{\rho}_+^A = \hat{\rho}_-^B &= \cos^2\frac{\theta}{2}|+\rangle\langle+| + \sin^2\frac{\theta}{2}|-\rangle\langle-| \\ \hat{\rho}_-^A = \hat{\rho}_+^B &= \sin^2\frac{\theta}{2}|+\rangle\langle+| + \cos^2\frac{\theta}{2}|-\rangle\langle-| \end{aligned} \tag{12}$$

Entanglement entropies

$$S_\pm(\theta) = \cos^2\frac{\theta}{2}\log(\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\log(\sin^2\frac{\theta}{2}) \tag{13}$$

Minimum entanglement for $\theta = 0$ ($\lambda/\omega = 0$), with $S_\pm(0) = 0$, maximum entanglement for $\theta = \pm\pi/2$ ($\omega/\lambda = 0$), with $S_\pm(0) = \log 2$. This is identical to the maximum possible entanglement entropy in the two-spin system.

PROBLEM 2

a) Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{14}$$

In the Heisenberg picture

$$\dot{\hat{a}}_H = \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right]_H = -i\omega_0\hat{a}_H - i\lambda e^{-i\omega t}\mathbb{1} \tag{15}$$

gives

$$\ddot{\hat{a}}_H = \frac{i}{\hbar}\left[\hat{H}, \dot{\hat{a}}_H\right] + \frac{\partial\dot{\hat{a}}_H}{\partial t} = -\omega_0^2\hat{a}_H - \lambda(\omega_0 + \omega)e^{-i\omega t}\mathbb{1} \tag{16}$$

which gives $C = -\lambda(\omega_0 + \omega)$.

b) Assume

$$\hat{a}_H = \hat{a}e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\mathbb{1} \tag{17}$$

Differentiation gives

$$\ddot{\hat{a}}_H = -\omega_0^2 \hat{a} e^{-i\omega_0 t} - D(\omega^2 e^{-i\omega t} - \omega_0^2 e^{-i\omega_0 t}) \\ = -\omega_0^2 \hat{a}_H - (\omega^2 - \omega_0^2) D e^{-i\omega t} \tag{18}$$

which is of the form (16) with

$$D = \frac{\lambda}{\omega - \omega_0} \tag{19}$$

c) Time evolution

$$\begin{aligned} |\psi(0)\rangle &= |0\rangle\,, \quad \hat{a}|0\rangle = 0 \\ |\psi(t)\rangle &= \hat{\mathcal{U}}(t)|\psi(0)\rangle \end{aligned} \tag{20}$$

gives

$$\begin{aligned} \hat{a}|\psi(t)\rangle &= \hat{\mathcal{U}}(t)\hat{\mathcal{U}}^\dagger(t)\,\hat{a}\,\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\;\hat{a}_H(t)\,|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\;(\hat{a}e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\,|\psi(0)\rangle \\ &= \frac{\lambda}{\omega - \omega_0}\,(e^{-i\omega t} - e^{-i\omega_0 t})\,|\psi(t)\rangle \end{aligned} \tag{21}$$

This shows that $|\psi(t)\rangle$ is a coherent state with time dependent complex parameter $z(t)$, and with real part $x(t)$, given by

$$z(t) = \frac{\lambda}{\omega - \omega_0}(e^{-i\omega t} - e^{-i\omega_0 t})\,, \quad x(t) = \frac{\lambda}{\omega - \omega_0}(\cos\omega t - \cos\omega_0 t) \tag{22}$$

The time evolution of the coordinate $x(t)$ is the same as for the classical driven harmonic oscillator,

$$\ddot{x} + \omega_0^2 x = -\lambda(\omega_0 + \omega)\cos\omega t \tag{23}$$

PROBLEM 3

a) Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{24}$$

Action on the states $|-,1\rangle$ and $|+,0\rangle$,

$$\begin{aligned} \hat{H}|-,1\rangle &= \frac{1}{2}\hbar\,(\omega|-,1\rangle + \lambda|+,0\rangle) \\ \hat{H}|+,0\rangle &= \frac{1}{2}\hbar\,(\omega|+,0\rangle + \lambda|-,1\rangle) \end{aligned} \tag{25}$$

Matrix form

$$\hat{H} = \frac{1}{2}\hbar\omega\mathbb{1} + \frac{1}{2}\hbar\lambda\sigma_x \tag{26}$$

Eigenvalues for σ_x are ± 1, gives energy eigenvalues

$$E_\pm = \frac{1}{2}\hbar(\omega \pm \lambda) \tag{27}$$

Energy eigenstates

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|-,1\rangle \pm |+,0\rangle)\,, \quad \hat{H}|\psi_\pm\rangle = E_\pm|\psi_\pm\rangle \tag{28}$$

Time dependent state

$$|\psi(t)\rangle = c_+ e^{-\frac{i}{\hbar}E_+ t}|\psi_+\rangle + c_- e^{-\frac{i}{\hbar}E_- t}|\psi_-\rangle \tag{29}$$

Initial condition $|\psi(0)\rangle = |-,1\rangle$ implies $c_+ = c_- = \frac{1}{\sqrt{2}}$,

$$|\psi(t)\rangle = e^{-\frac{i}{2}t}(\cos(\frac{\lambda}{2}t)|-,1\rangle - i(\sin(\frac{\lambda}{2}t)|+,0\rangle) \tag{30}$$

which gives $\epsilon = -\omega/2$ and $\Omega = \lambda/2$.

b) The Lindblad equation gives for the occupation probability of the ground state

$$\frac{dp_g}{dt} = -\frac{i}{\hbar}\langle -,0|\left[\hat{H},\hat{\rho}\right]|-,0\rangle + \gamma\langle -,0|\hat{a}\hat{\rho}\hat{a}^\dagger|-,0\rangle = \gamma\langle -,1|\hat{\rho}|-,1\rangle \tag{31}$$

When a photon is present in the cavity, $\langle -,1|\hat{\rho}|-,1\rangle \neq 0$, this gives $\dot{p}_g > 0$, which implies that the occupation probability of the ground state increases until there is no photon in the cavity, $\langle -,1|\hat{\rho}|-,1\rangle = 0$.

c) Evaluation of the matrix elements of the Lindblad equation in the subspace spanned by $|-,1\rangle$ and $|+,0\rangle$ gives

$$\begin{aligned}
\dot{p}_1 &= -\frac{i}{2}\lambda(\langle +,0|\hat{\rho}|-,1\rangle - \langle -,1|\hat{\rho}|+,0\rangle) - \gamma p_1 \\
\dot{p}_0 &= -\frac{i}{2}\lambda(\langle -,1|\hat{\rho}|+,0\rangle - \langle +,0|\hat{\rho}|-,1\rangle) \\
\dot{b} &= -\frac{i}{2}\lambda(\langle +,0|\hat{\rho}|+,0\rangle - \langle -,1|\hat{\rho}|-,1\rangle) - \frac{1}{2}\gamma b
\end{aligned} \tag{32}$$

which simplifies to

$$\begin{aligned}
\dot{p}_1 &= -\gamma p_1 - \lambda b \\
\dot{p}_0 &= \lambda b \\
\dot{b} &= -\frac{1}{2}\gamma b + \frac{1}{2}\lambda(p_1 - p_0)
\end{aligned} \tag{33}$$

Expected time evolution: Exponentially damped oscillations between the states $|-,1\rangle$ and $|+,0\rangle$, with the system ending in the photon less ground state $|-,0\rangle$.