FYS 4110/9110 Modern Quantum Mechanics

Midterm Exam, Fall Semester 2015

Return of solutions
The problem set is available from Monday morning, October 19.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday October 26, at 12:00.
Use candidate numbers rather than full names.

Language
Note: The problem set is available also in Norwegian.
Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems
Please ask the lecturer, Jon Magne Leinaas (Office: room 471Ø), or the assistant Ola Liabøtrø (room 469Ø).

The problem set consists of 2 problems written on 4 pages.

PROBLEMS

1 A three-spin problem

We consider a system consisting of three electrons. They all sit at fixed positions, with their spins as free variables.

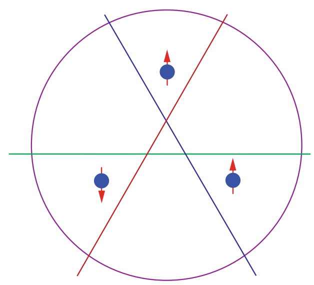

Figure 1: The three-spin-half system. Each of the straight lines shows a division of the full system into two parts, where one part contains a single spin and the othe part contains two spins.

a) The total spin we write as $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2 + \hat{\mathbf{S}}_3$. Use the rule for composition of quantum spins to show that the (spin) Hilbert space consists of three orthogonal subspaces, characterized by spin values $s = 1/2$, $1/2$ and $3/2$ respectively, with $\hat{\mathbf{S}}^2 = s(s+1)\hbar^2$.

b) We consider the following three states of the spin system

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}((|udd\rangle + e^{2\pi in/3}|dud\rangle + e^{-2\pi in/3}|ddu\rangle)\,, \quad n = 0, \pm 1 \tag{1}$$

where $|u\rangle$ is a spin up state along the z-axis and $|v\rangle$ is a spin down state along the same axis. We use the notation $|udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$, with the first factor in the tensor produce referring to particle 1, the second one to particle 2, and the last one to particle 3. Show that the vectors (1) are orthogonal and have well defined values for the the total spin operators $\mathbf{S}^2$ and S_z. Determine these values.

c) The three-particle system can be considered as a bipartite system, with particle 1 defining one subsystem and particles 2 and 3 defining the other part. We write this partition of the system symbolically as $123 = 1 + (23)$. With this partition what is the corresponding entanglement entropy of the system in the three cases $n = 0, \pm 1$? Compare with the maximum possible entanglement entropy in the bipartite system. With the two other partitions, $123 = 2 + (13)$ and $123 = 3 + (12)$, is there any difference in the entanglement?

d) A measurement of the observable $\hat{S}_{1z}$ is made on particle 1, with the system in one of the states $|\psi_n\rangle$. If the result is *spin up*, what is the entanglement of 2 and 3 in subsystem (23), after the measurement? If the result instead is *spin down*, what is then the entanglement?

e) Consider next the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle) \tag{2}$$

Determine the entanglement entropy of this state with respect to any of the partitions defined in c), and compare with the result found for the states $|\psi_n\rangle$.

We introduce state vectors for spin up and down in the x-direction by

$$|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)\,, \quad |b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle) \tag{3}$$

and for up and down in the y-direction by

$$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)\,, \quad |l\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle) \tag{4}$$

f) Rewrite the state vector (2) in two different ways, first by using the spin basis (3) for all three spins and next by using spin basis (4) for spin 1 and 2 and basis (3) for spin 3. Use the expressions to show that all spin components of particle 1, S_{1x}, S_{1y} and S_{1z}, can be determined by performing spin measurements on particles 2 and 3, while *not making any measurement* on particle 1. Specify in each case which measurement that should be performed on particle 2 and 3.

2 Entanglement and Bell inequalities

We consider an experimental situation, similar to the one discussed in the lecture notes, where pairs of spin 1/2 particles are initially prepared in a correlated spin state, and then are separated in space while keeping the spin state unchanged. When far apart spin measurements are performed on the particles in each pair, and the results are registered and compared.

The situation is illustrated in the figure, where a series of entangled pairs are created in a source K, and where measurements of the z-components of the spin are performed on both particles (A and B). When the spins in the z-directions are strictly anticorrelated, the result *spin up* (*spin down*) for particle A is always followed by the result *spin down* (*spin up*) for particle B.

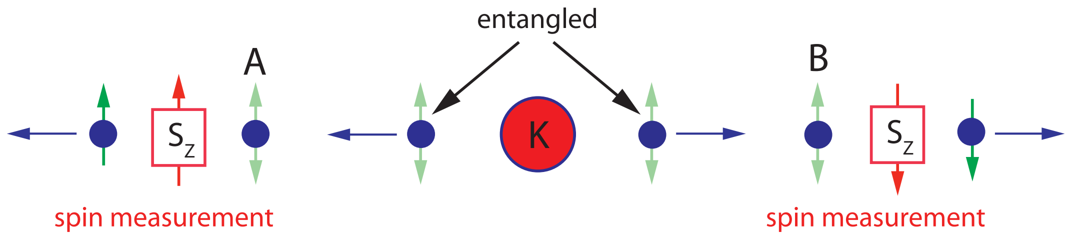

Figure 2: EPR experiment with correlated spins

We consider the situation where three different sets of measurements are performed, with different spin states,

$$\begin{aligned}
\text{I}: \quad & \hat{\rho}_1 = |\psi_a\rangle\langle\psi_a|, \quad |\psi_a\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \\
\text{II}: \quad & \hat{\rho}_2 = |\psi_s\rangle\langle\psi_s|, \quad |\psi_s\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\
\text{III}: \quad & \hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)
\end{aligned} \tag{5}$$

The notation is $|+-\rangle = |+\rangle \otimes |-\rangle$, where $|\pm\rangle$ are spin states of a single particle, with S_z quantized. The first factor in the tensor product refers to particle A and the second one to particle B. Note that all three states are strictly anticorrelated with respect to the z-component of the spin of the two particles. The purpose of the (hypothetical) experiment is to examine correlation functions that are relevant for the Bell inequalities, as already discussed for case I in the lecture notes, to see if the three states show different behavior. This involves performing the spin measurements also for rotated directions of the spin axes.

a) Of the three density operators only $\hat{\rho}_1$ is rotationally invariant. Demonstrate this by calculating the expectation value of $\mathbf{S}^2$ for the three cases, where $\mathbf{S} = (\hbar/2)(\boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{\sigma})$ is the spin vector of the full system, and comment on the results.

b) What are the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$ in the three cases? Determine the von Neumann entropy S of the full system, as well as the entropies S_A and S_B of the subsystems. Check if the classical restriction on the entropies $S \geq \max\{S_A, S_B\}$ is satisfied in any of the cases. In each of the cases examine if the states are entangled or separable, and give, if possible, a numerical measure of the degree of entanglement.

We assume the direction of the two measurement devices can be rotated so they measure spin components of the form

$$S_\theta = \cos\theta S_z + \sin\theta S_x \tag{6}$$

where the angle θ can be chosen independently for A and B. The state $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle$ is the *spin up* vector in the rotated direction and the operator $\hat{P}(\theta) = |\theta\rangle\langle\theta|$ projects on the corresponding spin vector.

c) Show that the given expression for $|\theta\rangle$, as claimed above, is the *spin up* state of S_θ. Determine the expectation value $P_A(\theta) = \left\langle \hat{P}(\theta) \right\rangle_A$, for particle A, in the three cases I, II and III. Comment on the result.

d) Determine, for the three cases, the joint probability distribution $P(\theta,\theta') = \left\langle \hat{P}(\theta) \otimes \hat{P}(\theta') \right\rangle$, with the two angles θ and θ' as independent variables.

The Bell inequality, according to the *hidden variable* analysis described in the lecture notes, gives a constraint on the possible classical correlations of the two spins. In the present case the inequality can be written as

$$F(\theta,\theta') \equiv P(0,\theta') - |P(\theta,0) - P(\theta,\theta')| \geq 0 \tag{7}$$

where one of the angles is set to 0 since we, for the states we consider, will only have strict anticorrelation for spin measurements along the z-axis. (For details see the derivation in the lecture notes.)

e) Make plots of the function $F(\theta, 0.5\,\theta)$ for the three cases I, II and III, with θ varying in the interval $0 < \theta < 2\pi$. Check in all cases whether the inequality (7) is satisfied or broken, and compare the results with what is known from point b) concerning entanglement between the two particles.

In addition to these plots, examine the functions for other choices $\theta' = \lambda\,\theta$ with $\lambda \neq 0.5$ to see if the results are not changed. Alternatively make a 3D plot of the two-variable function $F(\theta,\theta')$ and check whether the conclusion concerning the Bell inequality holds in the full parameter space.

f) Assume an experimental series is performed, with the two angles fixed. The number of pairs registered with *spin up* (in the chosen directions) for both spins A and B is n_{++}, and the number with *spin down* for both spins is n_{--}. Similarly n_{+-} is the number of pairs registered with *spin up* for A and *spin down* for B, n_{-+} is the number of pairs registered with *spin down* for A and *spin up* for B. The total number of pairs in the series is N.

We refer to the experimental results corresponding to $P_A(\theta)$, $P_B(\theta')$, and $P(\theta,\theta')$ as $P^A_{exp}(\theta)$, $P^B_{exp}(\theta')$, and $P_{exp}(\theta,\theta')$. What are these quantities expressed in terms of the numbers $\{n_{ij}, i,j = \pm\}$ and N?