

## FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2015

### Problem set 1

We begin the weekly sets with some problems concerning basic and useful mathematical relations.

#### 1.1 Commutators and anti-commutators

We use the standard notation for commutators and anticommutators

$$[A, B] = AB - BA \quad \{A, B\} = AB + BA \quad (1)$$

where  $A$  and  $B$  are two operators or matrices. Show the following identities,

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C] \\ [A, BC] &= \{A, B\}C - B\{A, C\} \end{aligned} \quad (2)$$

#### 1.2 Trace and determinant

We remind you about the following relations

$$\text{Tr}(AB) = \text{Tr}(BA), \quad \det(AB) = \det A \det B \quad (3)$$

a) Assume  $\hat{A}$  to be a quantum observable and  $A$  to be the matrix representation of the observable in an orthonormalized basis  $\{|n\rangle\}$ , which means

$$A_{mn} = \langle m | \hat{A} | n \rangle \quad (4)$$

We define the trace and determinant of the (abstract) operator as

$$\text{Tr } \hat{A} = \text{Tr } A, \quad \det \hat{A} = \det A \quad (5)$$

Show that if we change to a new basis  $\{|n'\rangle\}$ , which is related to the first by a unitary transformation, that will not change the values of the trace and determinant.

b) Assume a  $\hat{A}$  is a hermitian operator with eigenvalues  $a_n, n = 1, 2, \dots$ . Explain why the trace and determinant can be expressed in terms of the eigenvalues as

$$\text{Tr } \hat{A} = \sum_n a_n \quad \det \hat{A} = \prod_n a_n \quad (6)$$

c) The *spectral decomposition* of an hermitian operator  $\hat{A}$  is a sum of the form

$$\hat{A} = \sum_n a_n |n\rangle \langle n| \quad (7)$$

where  $a_n$  are the eigenvalues and  $|n\rangle$  are the corresponding eigenvectors of the operator. A function  $f(a)$  defines an *operator function*  $\hat{f} \equiv f(\hat{A})$  of  $\hat{A}$  by the related decomposition

$$\hat{f} \equiv \sum_n f(a_n) |n\rangle \langle n| \quad (8)$$

Use this definition and the results of problem b) to show that we have the following relation

$$\det e^{\hat{A}} = e^{\text{Tr } \hat{A}} \quad (9)$$

We assume the trace of  $\hat{A}$  to be well defined and finite (which may not necessarily be the case in an infinite dimensional Hilbert space).

d) Show that for general state vectors  $|\psi\rangle$  and  $|\phi\rangle$  we have the relation

$$\langle\psi|\phi\rangle = \text{Tr}(|\phi\rangle\langle\psi|) \quad (10)$$

### 1.3 Dirac's delta function

The basic relation defining the delta functions is the following

$$f(x) = \int_{-\infty}^{\infty} dx' \delta(x - x') f(x') \quad (11)$$

with  $f(x)$  as any chosen function. Clearly  $\delta(x)$  is not a function in the usual sense, and in particular it has the property that  $\delta(x) = 0$  for  $x \neq 0$  and  $\delta(0) = \infty$ . Nevertheless it is possible (with some care) to treat it as a function and as we know from the wave function description of quantum physics it is in many cases a very useful concept.

a) We remind you about the formulas for Fourier transformation in one dimension

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx} \quad (12)$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \quad (13)$$

Show that the delta function has the following Fourier expansion

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \quad (14)$$

b) Assume  $g(x)$  is a differentiable function with zero at one point  $x_0$ ,

$$g(x_0) = 0 \quad \text{for} \quad (15)$$

Assume also that the derivative does not vanish at this point,  $g'(x_0) \neq 0$ . Show by use of the definition (11) that we have the following relation

$$\delta(g(x)) = \frac{1}{|g'(x_0)|} \delta(x - x_0) \quad (16)$$

Assume that the function  $g(x)$  has several zeros, at the points  $x = x_i$ . Explain why this gives the generalized formula

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i) \quad (17)$$

### 1.4 Position and momentum eigenstates

The position and momentum eigenstates are given by the relations

$$\hat{x}|x\rangle = x|x\rangle \quad \langle x|x'\rangle = \delta(x - x') \quad (18)$$

$$\hat{p}|p\rangle = p|p\rangle \quad \langle p|p'\rangle = \delta(p - p') \quad (19)$$

Furthermore, in the x-representation the momentum operator is given by  $\hat{p} = -i\hbar \frac{d}{dx}$ . Use these relations together with the Fourier expansion of the delta function to show that the scalar product of a momentum and a position state is give by

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}xp} \quad (20)$$