FYS 4110 Modern Quantum Mechanics, Fall Semester 2015

Problem set 10

10.1 The canonical commutation relations.
The basic commutation relations of the electromagnetic field can be written as

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^{\dagger}_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{1}$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field and a is a polarization index. The photon creation and annihilation operators are related to the field operators by

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(\hat{a}_{\mathbf{k}a} + \hat{a}^{\dagger}_{-\mathbf{k}\bar{a}})\,, \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}(\hat{a}_{\mathbf{k}a} - \hat{a}^{\dagger}_{-\mathbf{k}\bar{a}}) \tag{2}$$

where $\bar{a}$ is related to a by a permutation of the two values of the polarization index. Use the above expressions to show that the creation and annihilation operators satisfy the standard harmonic oscillator commutation relations,

$$\left[\hat{a}_{\mathbf{k}a}, \hat{a}^{\dagger}_{\mathbf{k}'a'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{3}$$

10.2 The electromagnetic field energy and momentum.
The classical expressions for the electromagnetic field energy $\mathcal{E}$ is

$$\mathcal{E} = \frac{1}{2}\int d^3r(\epsilon_0\mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2) \tag{4}$$

The same expressions is valid in the quantum description, when the classical fields are replaced by the operator fields, $\mathbf{E} \to \hat{\mathbf{E}}$, $\mathbf{B} \to \hat{\mathbf{B}}$. The energy $\mathcal{E}$ is then replaced by the hamiltonian $\hat{H}$.

Use expressions for the field operators from the lecture notes to show that $\hat{H}$ has the following form, when written in terms of the photon number $\hat{N}_{\mathbf{k}a}$ operator

$$\hat{H} = \sum_{\mathbf{k}a}\hbar\omega_k\hat{N}_{\mathbf{k}a} + E_0\mathbb{1} \tag{5}$$

where $\hat{N}_{\mathbf{k}a}$ is defined by

$$\hat{N}_{\mathbf{k}a} = \hat{a}^{\dagger}_{\mathbf{k}a}\hat{a}_{\mathbf{k}a} \tag{6}$$

Comment on the interpretation of the constant E_0 and explain the meaning of removing this from the definition of the Hamiltonian, as one usually does.

10.3 Dressed photon states (Exam 2011)
A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are

admixtures of the photon and atomic states. These are sometimes referred to as a "dressed" photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{7}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{8}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the 2×2 identity matrix. Express the parameters Δ, $\cos\phi$, $\sin\phi$, and ϵ in terms of ω_0, ω and λ.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of ϕ, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.