FYS 4110 Modern Quantum Mechanics, Fall Semester 2015

Problem set 11

11.1 Photon emission

A particle with mass m and charge e is trapped in a one-dimensional harmonic oscillator potential, with the motion restricted to the z-axis. The frequency of the oscillator is ω. At time $t = 0$ the particle is excited to energy level n and it then performs a transition to level $n-1$ by emitting one photon of energy $\hbar\omega$. We write the energy eigenstates of the composite system of charged particle and photons as $|n, n_{\mathbf{k}a}\rangle$. With initially no photon present the state is $|i\rangle = |n, 0\rangle$, while the final state with one photon present is $|f\rangle = |n-1, 1_{\mathbf{k}a}\rangle$. To first order in perturbation theory the angular probability distribution $p(\theta, \phi)$ of the emitted photon is

$$p(\theta, \phi) = \kappa \sum_a |\langle n-1, 1_{\mathbf{k}a}|\hat{H}_{emis}|n, 0\rangle|^2 \tag{1}$$

with (θ, ϕ) as the polar angle of the photon quantum number $\mathbf{k}$ and κ as a proportionality factor. $\hat{H}_{emis}$ is the emission part of the interaction Hamiltonian, which in the dipole approximation is

$$H_{emis} = -\frac{e}{m} \sum_{\mathbf{k}a} \sqrt{\frac{\hbar}{2V\epsilon_0\,\omega}}\, \hat{\mathbf{p}} \cdot \boldsymbol{\varepsilon}_{\mathbf{k}a}\, \hat{a}^\dagger_{\mathbf{k}a} \tag{2}$$

a) Determine the particle matrix element $\langle n-1|\hat{\mathbf{p}}|n\rangle$.

b) Find the probability distribution $p(\theta, \phi)$.

11.2 Squeezed states

The definition we have used for the coherent states of a harmonic oscillator shows that these states depend on the frequency ω of the oscillator. It follows from the fact that the raising and lowering operators $\hat{a}^\dagger$ and $\hat{a}$, when expressed in terms of $\hat{x}$ and $\hat{p}$, are frequency dependent, while $\hat{x}$ and $\hat{p}$ are independent of ω.

In this problem we examine this dependence on the frequency by assuming that a particle in a harmonic oscillator potential, with initial frequency ω_a, is at time $t = 0$ is in a coherent state $|z_0\rangle_a$. At this moment there is a sudden change in the potential to a new frequency ω_b. The quantum state has no time to adjust to this abrupt change, so the state is $|z_0\rangle_a$ also immediately after the frequency has changed.

a) The change of frequency means that the raising and lowering operators are changed. We refer to the operators before the change as $\hat{a}^\dagger$ and $\hat{a}$ and as $\hat{b}^\dagger$ and $\hat{b}$ after the change. Show that we have the following relations between the operators

$$\hat{a} = c\,\hat{b} + s\,\hat{b}^\dagger\,, \quad \hat{a}^\dagger = s\,\hat{b} + c\,\hat{b}^\dagger\,, \tag{3}$$

with the inverse

$$\hat{b} = c\,\hat{a} - s\,\hat{a}^\dagger\,, \quad \hat{b}^\dagger = -s\,\hat{a} + c\,\hat{a}^\dagger\,, \tag{4}$$

and find c and s expressed in terms of the two frequencies ω_a and ω_b. Explain why we may assume c and s to represent hyperbolic functions of the form $c = \cosh\xi$, $s = \sinh\xi$, for some variable ξ.

b) Show that the two sets operators can be related by a unitary transformation

$$\hat{U} = e^{\frac{\xi}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} = e^{\frac{\xi}{2}(\hat{b}^2 - \hat{b}^{\dagger 2})} \tag{5}$$

so that

$$\hat{a} = \hat{U}\,\hat{b}\,\hat{U}^\dagger\,, \hat{a}^\dagger = \hat{U}\,\hat{b}^\dagger\,\hat{U}^\dagger \tag{6}$$

c) We now have two sets of coherent states defined by $\hat{a}|z\rangle_a = z|z\rangle_a$ and $\hat{b}|z\rangle_b = z|z\rangle_b$. Show that $|z_0\rangle_a$ is not a coherent state with respect to the new lowering operator $\hat{b}$.

d) We next consider the case $z_0 = 0$, so the initial state is the ground state of the Hamiltonian *before* the change of frequency. Show, by use of results from b), that this state, when expanded in the energy basis *after* the change, has the form

$$|0\rangle_a = \sum_{n=0}^{\infty} \alpha_n |2n\rangle_b \tag{7}$$

with only contributions from even numbers of excitations. Use the relations between the a and b operators to find a recursion formula for the expansion coefficients, and show that we have

$$\alpha_n = (-\frac{s}{2c})^n \frac{\sqrt{(2n)!}}{n!}\alpha_0 \tag{8}$$

e) In the original coherent state representation the state $|0\rangle_a$ is represented as the wave function $\psi_0(z) = {}_a\langle z|0\rangle_a$, and in the later coherent state representation as $\phi_0(z) = {}_b\langle z|0\rangle_a$. Use the above expansion to find the function $\phi_0(z)$ expressed as a sum over n. Choose the numerical value $s = \frac{1}{2}$, with the corresponding value for c, and make a numerical 3D or contour plot of the absolute value $|\phi_0(z)|^2$, with the real and imaginary components of z as variables. Compare with a similar plot of $|\psi_0(z)|^2$, and give an explanation for what we mean by referring to $\psi_0(z)$ as a *squeezed state*.

f) When time evolves for $t > 0$ the state $\psi_0(z)$ will rotate in the complex z-plane. Show that this is the case, and comment on what is the frequency of rotation.