FYS 4110 Modern Quantum Mechanics, Fall Semester 2015

Oppgavesett 12

12.1 Electric dipole transition in hydrogen (Exam 2008)

We consider the transition in hydrogen from the excited 2p level to the ground state 1s, where a single photon is emitted. The initial atomic state (A) we assume to have $m = 0$ for the z-component of the orbital angular momentum, so that the quantum numbers of this state are $(n, l, m) = (2, 1, 0)$, with n as the principle quantum number and l as the orbital angular momentum quantum number. Similarly the ground state (B) has quantum numbers $(n, l, m) = (1, 0, 0)$. When expressed in polar coordinates the wave functions of the two states (with intrinsic spin of the electron not included) are given by

$$\psi_A(r, \phi, \theta) = \frac{1}{\sqrt{32\pi a_0^3}} \cos\theta \, \frac{r}{a_0} \, e^{-\frac{r}{2a_0}}$$
$$\psi_B(r, \phi, \theta) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} \tag{1}$$

where a_0 is the Bohr radius.

We remind you about the form of the interaction matrix element in the dipole approximation,

$$\langle B, 1_{\mathbf{k}a} | \hat{H}_{emis} | A, 0 \rangle = ie \sqrt{\frac{\hbar\omega}{2V\epsilon_0}} \, \boldsymbol{\epsilon}_{\mathbf{k}a} \cdot \mathbf{r}_{BA} \tag{2}$$

where e is the electron charge, $\mathbf{k}$ is the wave vector of the photon, a is the polarization quantum number, ω is the photon frequency and $\boldsymbol{\epsilon}_{\mathbf{k}a}$ is a polarization vector. V is a normalization volume for the electromagnetic wave functions, ϵ_0 is the permittivity of vacuum and $\mathbf{r}_{BA}$ is the matrix element of the electron position operator between the initial and final atomic states.

a) Explain why the x- and y-components of $\mathbf{r}_{BA}$ vanish while the z-component has the form $z_{BA} = \nu a_0$, with ν as a numerical factor. Determine the value of ν. (A useful integration formula is $\int_0^\infty dx \, x^n \, e^{-x} = n!$.)

b) To first order in perturbation theory the interaction matrix element (2) determines the direction of the emitted photon, in the form of a probability distribution $p(\phi, \theta)$, where (ϕ, θ) are the polar angles of the wave vector $\mathbf{k}$. Determine $p(\phi, \theta)$ from the above expressions.

c) The life time of the 2p state is $\tau_{2p} = 1.6 \cdot 10^{-9} s$ while the excited 2s state (with angular momentum $l = 0$) has a much longer life time, $\tau_{2s} = 0.12 s$. Do you have a (qualitative) explanation for the large difference?

12.2 Radiation damping (Exam 2014)

A charged particle is oscillating in a one-dimensional harmonic oscillator potential. It emits electric dipole radiation, with the rate for transition between an initial state i and a final state f given by the radiation formula

$$\mathcal{W}_{fi} = \frac{4\alpha}{3c^2} \omega_{fi}^3 |x_{fi}|^2 \tag{3}$$

where α is the fine structure constant, $\hbar\omega_{fi}$ is the energy radiated in the transition, and c is the speed of light. x is the position coordinate of the particle, which is related to the raising and lowering operators of the harmonic oscillator by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \tag{4}$$

with m as the mass of the particle.

a) Show that the non-vanishing transition rates are of the form

$$\mathcal{W}_{n-1,n} = \gamma n \tag{5}$$

with $n = 0, 1, 2, ...$ as referring to the energy levels of the harmonic oscillator, and γ as a constant decay parameter. Detemine γ.

The time evolution of the quantum state of the oscillating particle is described by the Lindblad equation in the following way

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[H_0, \hat{\rho}] - \frac{1}{2}\gamma\left[\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right] \tag{6}$$

with $\hat{\rho}$ as the density operator of the particle and H_0 as the harmonic oscillator Hamiltonian, without decay.

b) In the following we focus on the diagonal terms of the density matrix, $p_n = \rho_{nn} = \langle n|\hat{\rho}|n\rangle$, which define the occupation probabilities of the energy eigenstates. Show that they satisfy the equation

$$\frac{dp_n}{dt} = -\gamma(np_n - (n+1)p_{n+1}) \tag{7}$$

Explain why this is consistent with the expression (5) for the transition rate $\mathcal{W}_{n-1,n}$.

c) Show that Eq. (7) implies that the expectation value of the excitation energy

$$E = \langle H_0\rangle - \frac{1}{2}\hbar\omega \tag{8}$$

decays exponentially with time.