FYS 4110 Modern Quantum Mechanics, Fall Semester 2015

Problem set 13

13.1 Time evolution in a two-level system (Exam 2013)

The Hamiltonian of a two-level system (denoted A) is $\hat{H}_0 = (1/2)\hbar\omega\,\sigma_z$, with σ_z as the diagonal Pauli matrix. We refer to the normalized ground state vector as $|g\rangle$ and the exited state as $|e\rangle$. In reality the system is coupled to a radiation field (denoted S), and the excited state will therefore decay to the ground state under emission of a quantum of radiation. $\hat{\rho}$ denotes the reduced density operator of subsystem A. To a good approximation the time evolution of this system is described by the Lindblad equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[H_0, \hat{\rho}\right] - \frac{1}{2}\gamma\left[\hat{\alpha}^\dagger\hat{\alpha}\hat{\rho} + \hat{\rho}\hat{\alpha}^\dagger\hat{\alpha} - 2\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger\right] \tag{1}$$

with γ as the decay rate for the transition $|e\rangle \to |g\rangle$, $\hat{\alpha} = |g\rangle\langle e|$ and $\hat{\alpha}^\dagger = |e\rangle\langle g|$.

In matrix form, with $\{|e\rangle, |g\rangle\}$ as basis, we write the density matrix as $\hat{\rho}$

$$\hat{\rho} = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{2}$$

with p_e as the probability for the system to be in state $|e\rangle$ and p_g as the probability for the system to be in state $|g\rangle$.

a) Assume initially the two-level system, at time $t = 0$, to be in state $\hat{\rho} = |e\rangle\langle e|$. Show, by use of Eq. (1), that p_e decays exponentially, with γ as decay rate, while the total probability $p_e + p_g$ is conserved.

b) Assume next that the system is initially in the following superposition of the two eigenstates of $\hat{H}_0$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Determine the time dependent density matrix $\hat{\rho}(t)$ with this initial state.

c) The density operator of subsystem A can alternatively be expressed in terms of the Pauli matrices as $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$. Determine the function $r^2(t)$ in the two cases above and show that in both cases it has a minimum for $t = (1/\gamma)\ln 2$. What is the minimum value for r in the two cases? Comment on the implication the results give for the entanglement between the two subsystems A and S. (We assume A+S all the time to be in a pure state.)

13.2 A radiation problem (Exam 2011)

We consider a one-dimensional problem where a two-level system (A) interacts with a scalar radiation field (B). The notation we use is similar to that in Problem 10.3. The Hamiltonian of the system we consider is

$$\hat{H} = \frac{1}{2}\hbar\omega_A\,\sigma_z + \sum_k \hbar\omega_k\hat{a}_k^\dagger\hat{a}_k + \kappa\sum_k\sqrt{\frac{\hbar}{2L\omega_k}}(\hat{a}_k\sigma_+ + \hat{a}_k^\dagger\sigma_-) = \hat{H}_0 + \hat{H}_{int} \tag{3}$$

The first term is the two-level Hamiltonian, with energy splitting $\hbar\omega_A$, the second one is the free field contribution, with $k = 2\pi n/L$ (n - integer) as the wave number of the photon. L is a (large) normalization length. The third term is the interaction term $\hat{H}_{int}$, with κ as an interaction parameter. The frequency parameter is $\omega_k = c|k|$.

a) A general state of the two-level system is characterized by a vector $\mathbf{r}$, with $r \leq 1$, and with the corresponding density matrix as

$$\rho_A = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \tag{4}$$

Consider first that the interaction term $\hat{H}_{int}$ is turned off, $\kappa = 0$, so that the time evolution operator of the two-level system is $\hat{\mathcal{U}}(t) = \exp(-\frac{i}{2}\omega_A \, t \, \sigma_z)$. Use this to determine the the density matrix $\rho_A(t)$ at time t, assuming that $\rho_A(0)$ is identical to the density matrix in (4), and show that the time evolution of $\mathbf{r}$ is a precession around the z-axis with angular velocity ω_A.

b) Assume next that $\kappa \neq 0$ and that initially the two-level system is in the excited "spin up state", while the scalar field is in the vacuum state. Thus, the initial state is $|+,0\rangle = |+\rangle \otimes |0\rangle$. It decays to the "spin down state" by emission of a field quantum. The final state we then write as $|-,1_k\rangle = |-\rangle \otimes |1_k\rangle$.

The occupation probability of the excited state $|+\rangle$ decays exponentially, $P_+(t) = \exp(-\gamma t)$, with a decay rate γ that to first order in the interaction, and in the limit $L \to \infty$, is given by

$$\gamma = \frac{L}{(2\pi\hbar)^2} \int dk \, |\langle -, 1_k | \hat{H}_{int} | +, 0\rangle|^2 \, \delta(\omega_k - \omega_A) \tag{5}$$

Determine the decay rate γ, expressed in terms of the parameters of the problem.

As discussed in the lectures an approximate way to handle the decay is to introduce an imaginary contribution to the energy of the decaying state. Assuming a more general initial state, of the form

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle) \otimes |0\rangle = \alpha|+,0\rangle + \beta|-,0\rangle \tag{6}$$

with α and β as unspecified coefficients, with $|\alpha|^2 + |\beta|^2 = 1$, we make the corresponding *ansatz* for the time evolved state

$$|\psi(t)\rangle = (e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle) \otimes |0\rangle + \sum_k c_k(t)|-,1_k\rangle \tag{7}$$

with $c_k(t)$ as decay parameters, which satify $c_k(0) = 0$.

c) Check what normalization of the state vector (7) means for the decay parameters, and determine the reduced density matrix matrix $\rho_A(t)$ of the two-level system.

d) Assume the same initial conditions as in b), $z(0) = 1$, $x(0) = y(0) = 0$ ($\alpha = 1$, $\beta = 0$). Determine the density matrix $\rho_A(t)$ and the corresponding time dependent vector $\mathbf{r}(t)$. Is the time evolution consistent with the expected exponential decay of the excited state of the two-level system? Give a brief description of the evolution of the entanglement between the two level system and the radiation field during the decay.

e) Choose another initial condition $x(0) = 1$, $y(0) = z(0) = 0$ ($\alpha = \beta = 1/\sqrt{2}$), and find also in this case the time evolution of the reduced density matrix and the components of the vector $\mathbf{r}(t)$. Sketch the time evolution of $\mathbf{r}(t)$ and compare qualitatively the motion with that in a) and d). Find $r(t)^2$ expressed as a function of γt, and sketch also this function. What does it show about the time evolution of the entanglement between the two subsystems A and B?

Assume in this paragraph $\gamma << \omega_A$.