FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2015

Problem set 2

2.1 Operator identities

Assume $\hat{A}$ and $\hat{B}$ to be two operators, generally not commuting.
We define the following to composite operators:

$$\begin{aligned} \hat{F}(\lambda) &= e^{\lambda \hat{A}} \hat{B} e^{-\lambda \hat{A}} \\ \hat{G}(\lambda) &= e^{\lambda \hat{A}} e^{\lambda \hat{B}} \end{aligned} \tag{1}$$

a) Show the following relation

$$\frac{d\hat{F}}{d\lambda} = \left[\hat{A}, \hat{F}\right] \tag{2}$$

and use it to derive the expansion

$$\hat{F}(\lambda) = \hat{B} + \lambda \left[\hat{A}, \hat{B}\right] + \frac{\lambda^2}{2} \left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] ... \tag{3}$$

b) Show the following relation between $\hat{G}(\lambda)$ and $\hat{F}(\lambda)$,

$$\frac{d\hat{G}}{d\lambda} = (\hat{A} + \hat{F})\hat{G} \tag{4}$$

and use this to demonstrate the following expansion (Campbell-Baker-Hausdorff)

$$\hat{G}(\lambda) = e^{\lambda \hat{A} + \lambda \hat{B} + \frac{\lambda^2}{2}[\hat{A}, \hat{B}] + \,...} \tag{5}$$

by calculating the exponent on the right-hand side to second order in λ.

c) When $\left[\hat{A}, \hat{B}\right]$ commutes with both $\hat{A}$ and $\hat{B}$ the expression (5) is exact without the higher order terms indicated by ... in (5).
Verify this by use of (3) and (4), and by noting that the eigenvalues of $\hat{G}$ satisfy a differential equation that can be integrated.

2.2 Gaussian integral

The following formula gives the integral of a gaussian function

$$I \equiv \int_{-\infty}^{\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \tag{6}$$

This is correct for complex λ provided the real part of λ is positive. Verify this by writing the I^2 as a two-dimensional integral

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, dy\, e^{-\lambda(x^2+y^2)} \tag{7}$$

and by changing to to polar coordinates in the evaluation.

2.3 Gaussian wave function

The quantum state of a free particle with mass m in one dimension is, at time $t = 0$, described by the following momentum wave function of Gaussian form,

$$\psi(p) = \sqrt[4]{\frac{\lambda}{\pi}}\, e^{-\frac{\lambda}{2}(p-p_0)^2} \tag{8}$$

where λ is a real, positive parameter that determines the width of the Gaussian. What is the corresponding time dependent wave function $\psi(x,t)$ in the coordinate representation? Show that this function is also a Gaussian, of the form

$$\psi(x,t) = N' e^{-\frac{\lambda'}{2}(x-x_0)^2} \tag{9}$$

with λ' and x_0 as a time-dependent parameters and N' as a normalization factor. Determine λ' and x_0 and examine how the maximum of the wave packet moves with time and how the width changes. Compare with the classical motion of the particle.

2.4 Time dependent transformations

Two unitarily equivalent descriptions of a quantum system are related by a *time dependent* unitary transformation $\hat{U}(t)$, which acts on state vectors as

$$|\psi(t)\rangle \quad \to \quad |\psi'(t)\rangle = \hat{U}(t)|\psi(t)\rangle \tag{10}$$

and on the observables as

$$\hat{A} \quad \to \quad \hat{A}'(t) = \hat{U}(t)\,\hat{A}\,\hat{U}(t)^{-1}\,. \tag{11}$$

Show that the Hamiltonian $\hat{H}'$, which determines the Schrödinger equation of the transformed state vector $|\psi'(t)\rangle$, includes an additional term which depends on the time derivative of $\hat{U}(t)$,

$$\hat{H} \to \hat{H}'(t) = \hat{U}(t)\,\hat{H}\,\hat{U}(t)^{-1} + i\hbar\frac{d\hat{U}}{dt}\hat{U}^{-1} \tag{12}$$

Discuss the meaning of the difference between the equations (11) and (12).

A unitary operator can generally be expressed as

$$\hat{U} = e^{i\hat{B}} \tag{13}$$

where $\hat{B}$ is a hermitian operator. Assume $\hat{B}$, and therefore also $\hat{U}$, is time dependent. Assume also that the following commutator vanishes, $\left[\left[\hat{B}, \frac{d\hat{B}}{dt}\right], \hat{B}\right] = 0$. Find the time derivative of $\hat{U}$ expressed in terms of $\hat{B}$ and its time derivative, and show that when introduced in Eq.(12), this equation gets the following form

$$\hat{H}' = e^{i\hat{B}}\,\hat{H}\,e^{-i\hat{B}} - \hbar\left(\frac{d\hat{B}}{dt} + \frac{i}{2}\left[\hat{B}, \frac{d\hat{B}}{dt}\right]\right) \tag{14}$$

Note in particular the contribution from the commutator between $\hat{B}$ and its time derivative.