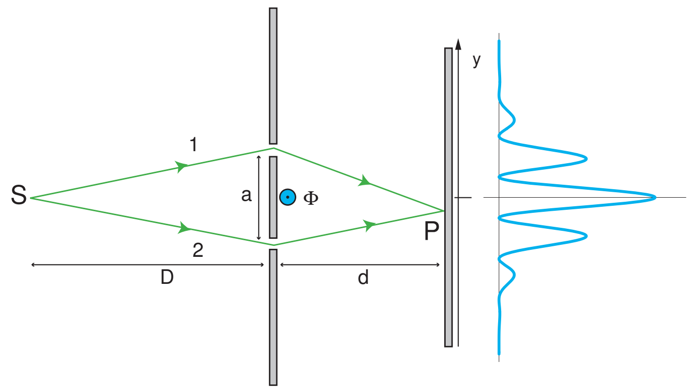
FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2015

Problem set 3

3.1 The Aharonov-Bohm effect

We consider a double slit experiment as sketched in the figure. Electrons are emitted from a source S and can pass trough one of the two slits of a first screen before being registered on a second screen. When a large number of electrons are registered they are found to form an interference patterns with minima and maxima on the screen.



Behind the middle part of the first screen a solenoid is placed which carries a magnetic flux Φ. The direction of the solenoid is parallel to the direction of the two slits, so that the paths through the upper slit pass on one side of the solenoid and the paths through the lower slit pass on the other side of the solenoid. We consider the magnetic field to be completely screened from the region where the electrons move, so that at no point along the trajectories of the electron there is a magnetic force acting on the particles. Nevertheless, quantum theory predicts that the strength of the magnetic flux will influence the interference pattern so that the maxima and minima are shifted up or down when the flux is changed. This is called the Aharonov-Bohm effect.

We consider in the following the distance d between the screens and the distance D between the source and the first screen to be much larger than the distance a between the two slits, and also to be much larger than the distance y from the central point of the second screen to any point P where an electron is registered.

As a reminder the classical Lagrangian of an electron moving in a magnetic field is

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 + e\mathbf{A}(\mathbf{r}) \cdot \dot{\mathbf{r}} \tag{1}$$

with $\mathbf{A}$ as the vector potential, and the magnetic field thus given as $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. As follows from Stokes' theorem the magnetic flux is given as the line integral

$$\Phi = \oint_C \mathbf{A} \cdot d\mathbf{r} \tag{2}$$

where C is any given closed loop that encircles once the solenoid.

We consider the situation where a single electron is emitted at time $t = 0$ from the source and is registered at a later time t at a point P of the screen. The probability distribution over the screen for where the electron is registered can be written as

$$p(y) = \lambda |\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0)|^2 \tag{3}$$

with y as the vertical coordinate of P, λ as a proportionality factor and $\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0)$ as the propagator from the initial point $(\mathbf{r}_S, 0)$ to the final point $(\mathbf{r}_P, t)$.

We consider in the following the the semi-classical approximation to the propagator, which we write as

$$\mathcal{G}(\mathbf{r}_P, t; \mathbf{r}_S, 0) = N \sum_{n=1}^{2} e^{\frac{i}{\hbar} S_n} \tag{4}$$

where S_n is the action integral for classical free-particle motion either through the upper slit ($n = 1$) or through the lower slit ($n = 2$), and N is a (y-dependent) normalization factor which is assumed to be independent of the path. Since the classical motion is not affected by the magnetic field, both λ and N are independent of the magnetic flux.

a) Show that the probability $p(y)$ depends on the *difference* between the action integrals of the two paths.

b) Show that the difference between the two action integrals can be written as a function of the magnetic flux Φ.

c) Show that the probability $p(y)$ depends periodically on the magnetic flux Φ. What is the flux period? Describe qualitatively how the interference pattern changes with variations in Φ.

3.2 Poisson summation

Consider an unspecified function $g(x)$ in one dimension. We define the related function $f(x)$ by

$$f(x) = \sum_{n=-\infty}^{\infty} g(x+n) \tag{5}$$

where we assume the infinite sum to be well defined.

a) Show that f is a periodic function, $f(x+1) = f(x)$, and therefore can be expressed as a discrete Fourier sum on the interval $0 \le x < 1$,

$$f(x) = \sum_{l=-\infty}^{\infty} c_l \, e^{2\pi i l x} \tag{6}$$

with coefficients

$$c_l = \int_{-\infty}^{\infty} g(x) e^{-2\pi i l x} dx \tag{7}$$

b) Show from this the identity

$$\sum_{n=-\infty}^{\infty} g(x+n) = \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} g(y) e^{2\pi i l (x-y)} dy \tag{8}$$

This expression is known as a Poisson summation, and when the integral can be performed it gives in many cases a useful resummation of the original sum.

c) Assume $g(x)$ to be specified as a Gaussian function

$$g(x) = ke^{-\lambda x^2} \tag{9}$$

with k and λ as a constants. Find the explicit expression for the identity (8) in this case. (Use the results for Gaussian integrals from one of the earlier problem sets, and consider λ to have a positive real part in order for the integral to converge.)

3.3 Jacobi's Theta Function

As a preparation for Problem 3.5 we consider here the symmetries of a special function called the *Jacobi Theta Function* θ_3. The function $\theta_3(z|w)$ is defined by the sum

$$\theta_3(z|w) = \sum_{l=-\infty}^{\infty} \exp(i\pi w l^2 + 2ilz) \tag{10}$$

a) Show that this sum has the same form as the Fourier sum (6) of the periodic Gaussian (derived from (9)), with $z = \pi x$, and show that the alternative expression for θ_3 that corresponds to the sum (5) is

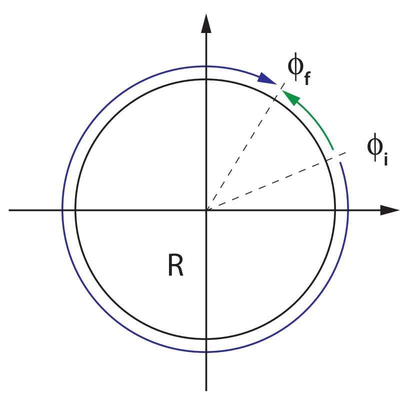
$$\theta_3(z|w) = \sqrt{\frac{i}{w}} \sum_{n=-\infty}^{\infty} \exp[-\frac{i}{\pi w}(z + \pi n)^2] \tag{11}$$

b) Show, by use of the above results, that function θ_3 has the following symmetry properties

$$\begin{aligned} \theta_3(z + \pi|w) &= \theta_3(z|w) \\ \theta_3(z + \pi w|w) &= e^{-i\pi w - 2iz}\theta_3(z|w) \\ \theta_3(z|w) &= \sqrt{\frac{i}{w}}\, e^{-iz^2/(\pi w)}\, \theta_3(\frac{z}{w}| - \frac{1}{w}) \end{aligned} \tag{12}$$

3.5 Particle on a circle

A particle with mass m moves freely on a circle of radius R.



a) Use the polar angle ϕ as coordinate and find the expression for the angular momentum eigenstates, $\psi_l(\phi) = \langle\phi|l\rangle$, with l as the dimensionless angular momentum quantum number (which take integer values). These states are also energy eigenstates. What is the energy E_l, expressed in terms of of l?

b) Find an expression for the propagator $\mathcal{G}(\phi, t; 0, 0) = \langle\phi|\hat{\mathcal{U}}(t, 0)|0\rangle$ as a sum over angular momenta, by making a direct calculation of the relevant matrix element of the time evolution operator $\hat{\mathcal{U}}(t, 0)$. (The coordinates of the initial position are here chosen as $(\phi_i, t_i) = (0, 0)$.) Show that the propagator can be expressed in terms of the Jacobi theta function $\theta_3(z|w)$.

c) Explain why there is an infinity of classical paths, with different *winding numbers* n, that connect the two points $(0, 0)$ and (ϕ, t), and use the semi-classical expression for the path integral to write the propagator $\mathcal{G}(\phi, t; 0, 0)$ as a sum over winding numbers n. Show that also this sum can be expressed in terms of the Jacobi theta function.

d) Use the results of Problem 3.4 to show that expressions found for the propagator in b) and c) are equivalent. This demonstrates that the semi-classical expression for the propagator of a free particle on a circle, in the same way as for a free particle on a line, is identical to the exact expression for the propagator.