FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2015

Problem set 5

5.1 Ladder operators in the Heisenberg picture

Consider a harmonic oscillator with Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) \tag{1}$$

expressed in terms of the ladder operators $\hat{a}^\dagger$ and $\hat{a}$. Show, by use of relevant formula in Prob. 2.1, that these two operators take the following time dependent form in the Heisenberg picture

$$\hat{a}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger\,, \quad \hat{a}(t) = e^{-i\omega t}\hat{a} \tag{2}$$

5.2 Displacement operators in phase space

For a particle moving in one dimension the position coordinate x and the momentum p define the coordinates of the two-dimensional classical *phase space*.

In the quantum description of the one-dimensional harmonic oscillator non-hermitian lowering operator is defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\,\hat{x} + i\hat{p}) \tag{3}$$

We may consider this as the operator of a complex phase space variable, with position as the real part and momentum as the imaginary part. It has a dimensionless form due to the constants introduced in the expression.

A coherent state, in a similar way is characterized by a complex number z, the eigenvalue of $\hat{a}$, which we may interpret as a complex phase space coordinate,

$$z = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\,x_c + ip_c) \tag{4}$$

The following operator

$$\hat{\mathcal{D}}(z) = e^{(z\hat{a}^\dagger - z^*\hat{a})} \tag{5}$$

acts as a displacement operator in phase space, in the sense

$$\hat{\mathcal{D}}(z)^\dagger \hat{x}\hat{\mathcal{D}}(z) = \hat{x} + x_c\,, \;\; \hat{\mathcal{D}}(z)^\dagger \hat{p}\hat{\mathcal{D}}(z) = \hat{p} + p_c \tag{6}$$

Show that displacements in two different directions in general will not commute but rather satisfy a relation of the form

$$\hat{\mathcal{D}}(z_a)\hat{\mathcal{D}}(z_b) = e^{i\alpha(z_a,z_b)}\hat{\mathcal{D}}(z_b)\hat{\mathcal{D}}(z_a) \tag{7}$$

with $\alpha(z_a, z_b)$ as a complex phase. Determine the phase as a function of z_a and z_b. What is the condition for the two operators to commute?

5.3 Eigenvectors for $\hat{a}^\dagger$?

The coherent states $|z\rangle$ are defined as eigenvectors of the lowering operator $\hat{a}$. Assume $|\bar{z}\rangle$ to be eigenvector of the raising operator $\hat{a}^\dagger$,

$$\hat{a}^\dagger|\bar{z}\rangle = \bar{z}|\bar{z}\rangle \tag{8}$$

Show that no normalizable vector exists that satisfies this equation by expanding the state $|\bar{z}\rangle$ in the energy eigenstates $|n\rangle$.

5.4 Driven harmonic oscillator (Midterm Exam 2008)

The Hamiltonian of a one-dimensional harmonic oscillator is given by the expression

$$\hat{H} = \frac{1}{2m}(\hat{p}^2 + m^2\omega_0^2\hat{x}^2) = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{9}$$

with the *raising* and *lowering* operators defined by

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0\hat{x} + i\hat{p}) \; , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega_0}}(m\omega_0\hat{x} - i\hat{p}) \tag{10}$$

The time evolution operator is

$$\hat{\mathcal{U}}_0(t) = e^{-\frac{i}{\hbar}t\hat{H}_0} \tag{11}$$

The coherent states of the oscillator are defined as eigenvectors of the lowering operator

$$\hat{a}|z\rangle = z|z\rangle \tag{12}$$

The general coherent state $|z\rangle$ is related to the ground state of the oscillator $|0\rangle$ by

$$|z\rangle = \hat{\mathcal{D}}(z)|0\rangle = e^{-z^*z}e^{z\hat{a}^\dagger}|0\rangle \tag{13}$$

where the unitary shift operator is given by

$$\hat{\mathcal{D}}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}} \tag{14}$$

a) Show that for a general operator $\hat{A}$ we have the relation

$$\hat{\mathcal{U}}e^{\hat{A}}\hat{\mathcal{U}}^{-1} = e^{\hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^{-1}} \tag{15}$$

and use that to calculate the operator $\hat{\mathcal{U}}_0(t)\hat{\mathcal{D}}(z)\hat{\mathcal{U}}_0(t)^\dagger$. Make use of the result to determine the time dependent state vector $|\psi(t)\rangle$, when this initially is a coherent state $|\psi(0)\rangle = |z_0\rangle$. Show that $|\psi(t)\rangle$ at later times t is also a coherent state.

We next assume the harmonic oscillator to be under influence of a time dependent external potential, so that the hamiltonian now is

$$\hat{H} = \hat{H}_0 + \hat{W}(x,t) \tag{16}$$

In the following we assume the external potential to have the specific form

$$\hat{W}(x,t) = A\hat{x}\sin\omega t \tag{17}$$

with A as a constant and ω as the oscillation frequency of the external potential.

b) Find the Heisenberg equation of motion for $\hat{x}$ and $\hat{p}$ and show that they correspond to the equation of motion of a *driven* harmonic oscillator that is subject to the periodic force $f(t) = -A\sin\omega t$.

c) Give the definition of the time evolution operator $\hat{\mathcal{U}}_I(t)$ in the *interaction picture* (see Sect. 1.3.1 of the lecture notes) and show that it satisfies an equation of the form

$$i\hbar\frac{d}{dt}\hat{\mathcal{U}}_I(t) = \hat{H}_I(t)\hat{\mathcal{U}}_I(t) \tag{18}$$

Assume $\hat{W}$ is treated as the interaction. Show that $\hat{H}_I(t)$ then is a linear function of $\hat{a}$ and $\hat{a}^\dagger$,

$$\hat{H}_I(t) = \theta(t)^*\,\hat{a} + \theta(t)\,\hat{a}^\dagger \tag{19}$$

and determine the function $\theta(t)$.

d) Show that the equation (18) has a solution of the form

$$\hat{\mathcal{U}}_I(t) = e^{\xi(t)\hat{a}^\dagger - \xi^*(t)\hat{a}}\,e^{i\phi(t)} \tag{20}$$

with $\xi(t)$ as a complex and $\phi(t)$ as a real function of time. What are the equations that these two functions should satisfy?

e) Use the expressions for $\hat{\mathcal{U}}_0(t)$ and $\hat{\mathcal{U}}_I(t)$ to find the time dependent state vector $|\psi(t)\rangle$ in the Schrödinger picture, with the same initial condition as in a). Show that also in this case it describes a time dependent coherent state, of the form $|\psi(t)\rangle = e^{i\gamma(t)}|z(t)\rangle$. Find $z(t)$ expressed in terms of z_0, $\xi(t)$ and ω_0.

f) Determine the function $\xi(t)$ and find an explicit expression for $z(t)$. The corresponding real coordinate is $x(t) = \sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,z(t)$. Does this coordinate satisfy the classical equation of motion of the driven harmonic oscillator?