FYS 4110 Non-relativistic Quantum Mechanics, Fall Semester 2015

Problem set 8

8.1 Coupled two-level systems

Two coupled two-level systems A and B are described by the following Hamiltonian

$$\hat{H} = \frac{\epsilon}{2}(3\,\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

where the first factor in the tensor product refers to system A and the second factor to system B. In the equation we use the definition $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

a) Write the Hamiltonian as a 4x4 matrix and show that two of the eigenvalues and eigenvectors are independent of λ. Introduce new variables, defined by $\epsilon = \mu\cos\theta$ and $\lambda = \mu\sin\theta$. Solve the eigenvalue problem for the remaining two-dimensional subspace and determine both the energies and eigenvectors as functions of μ and θ.

b) Express the two eigenstates as 4x4 density matrices and determine the reduced density matrices for the two subsystems A and B.

c) Determine the entropy of the reduced density matrices as functions of and θ. For what parameter values is the entanglement of the two subsystems maximal?

8.2 Uncle Charlie's gift

As we have seen in this course, quantum physics can make some unexpected twists to what we normally consider as possible in the communication between two parties. The present problem is due to Jan Myrheim (NTNU).

The eccentric Uncle Charlie has declared his intention to give either his niece Alice (A) or his nephew Bob (B) a generous gift. He has informed them about this and also that the gift is either a million dollars or a new bicycle. In order to test their quantum physics abilities he has sent them one qubit each (by decoherence protected airmail), and has informed them that the two-qubit system is in one out of four possible states,

$$\begin{aligned}
|Aa\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)\\
|Ab\rangle &= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)\\
|Ba\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)\\
|Bb\rangle &= \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned} \tag{2}$$

with $|00\rangle = |0\rangle \otimes |0\rangle$ etc., where the first factor corrsponds to the qubit sent to Alice and the second factor to the qubit sent to Bob. The state of the two qubits contains the information about his decision, with Aa meaning that Alice will get one million dollars, Ab meaning she will get a bicycle, and with similar outcome for Bob when Ba or Bb has been chosen.

Charlie challenges them to find the information by making measurements on their qubits, but Alice and Bob are living far apart, Alice in Norway and Bob in Australia, and their communication is therefore restricted to a classical channel (telephone line) when they want to discuss how to perform the measurement.

After a discussion they reach the frustrating conclusion that they cannot obtain the full information about Uncle Charlie's decision, and they consider instead what is the best information they will be able to extract. The challenge for you is to make a similar analysis.

a) Alice and Bob first consider measuring the qubit states in the $\{|0\rangle, |1\rangle\}$ basis, but they decide that this will give them no information what so ever about Charlie's decision. Why is that the case?

b) At the next step Alice finds that it is better that she measures her qubit in the basis $\{|u\rangle, |v\rangle\}$, where

$$|u\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,, \quad |v\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{3}$$

while Bob makes the measurement in the original $\{|0\rangle, |1\rangle\}$ basis. Show that in such a measurement, the four possible outcomes of the measurements would give them the information restricted to the two possibilities, 1: Aa or Ba, 2: Ab or Bb. That means they will get the information about what the gift is but not about who will get this gift.

c) They also consider a measurement where Alice uses the $\{|0\rangle, |1\rangle\}$ basis, while Bob uses the $\{|u\rangle, |v\rangle\}$ basis. What is the information they can get in this way? They further consider the situation here both of them make the measurements in the $\{|u\rangle, |v\rangle\}$ basis. Can more information be extracted with this choice of measurements?

They now make a more complete analysis of the possible measurements by assuming Alice uses a general, orthogonal basis for her measurement,

$$|w\rangle = \alpha|0\rangle + \beta|1\rangle\,, \quad |x\rangle = -\beta^*|0\rangle + \alpha^*|1\rangle \tag{4}$$

with $|\alpha|^2 + |\beta|^2 = 1$, and Bob uses the basis

$$|y\rangle = \gamma|0\rangle + \delta|1\rangle\,, \quad |z\rangle = -\delta^*|0\rangle + \gamma^*|1\rangle \tag{5}$$

with $|\gamma|^2 + |\delta|^2 = 1$.

d) Show that by properly choosing the parameters α, β, γ and δ they will be able to extract the information restricted to the two possibilities, 1: Aa or Bb, 2: Ab or Ba. In this case the result 1 would then tell that either Alice gets a million dollars or Bob gets a bicycle and result 2 would tell them that either Bob gets a million dollars or Alice gets a bicycle. At the end they decide that this may be the most interesting information.

e) Explain why any of the measurements discussed above would erase the rest of the information from the qubits, so that a second measurement would not give any additional information about the gift.

Uncle Charlie's gift would in any case seem unfair and leave at least one of the two discontent. Let us hope that when he realizes that they both have obtained a good understanding of quantum physics through their studies he will compensate in some way the one that does not get the gift in such a way that they both will be happy with the situation.