FYS 4110 Modern Quantum Mechanics, fall semester 2015

Problem set 9

9.1 To-level systems (Exam 2009)

A quantum system is composed by two two-level systems, $\mathcal{A}$ and $\mathcal{B}$. The Hilbert space of the total system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is then of dimension four. The two subsystems are dynamically coupled, with the Hamiltonian of the full system having the form

$$\hat{H} = \frac{1}{2}\hbar\omega(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - i\hbar\lambda(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+) \tag{1}$$

with σ_z and $\sigma_\pm$ as Pauli matrices, and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$. $\hbar\omega$ is the splitting of the two energy levels of each of the two (uncoupled) systems, while λ is an interaction parameter for the composite system. In the tensor products we assume the first factor to act on subsystem $\mathcal{A}$ and the second factor to act on subsystem $\mathcal{B}$.

a) Show that the time dependent Schrödinger equation has a solution of the form

$$|\psi(t)\rangle = \cos(\lambda t)|+-\rangle + \sin(\lambda t)|-+\rangle \tag{2}$$

where $|+-\rangle = |+\rangle \otimes |-\rangle$ and $|-+\rangle = |-\rangle \otimes |+\rangle$, and where $\sigma_z|\pm\rangle = \pm|\pm\rangle$ for each of the subsystems. What is the expression for the corresponding density operator $\hat{\rho}(t)$, when this is written in bra-ket form?

b) The time dependent density operator can also be expressed in terms of Pauli matrices, in a similar way as in (1). Find this expression, and also find the reduced density operators $\hat{\rho}_A(t)$ og $\hat{\rho}_B(t)$, both expressed in terms of Pauli matrices (and the identity operator).

c) Give the general expression for the entanglement entropy of the composite system when the system is in a *pure* quantum state. For the particular, time dependent state (2), what is the expression?

9.2 Distributed information (Exam 2012)

A secret message is distributed to a party of three, denoted A, B, and C, in the form of an entangled three-spin state, coded into three spin-half particles. As the receiving party knows in advance, the quantum state is one out of a selection of three,

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}(|+--\rangle + \eta^n|-+-\rangle + (\eta^*)^n|--+\rangle)\,, \quad \eta = e^{2\pi i/3} \tag{3}$$

where $n = 0, 1, 2$. The message is identified by the value of n, which means by which of the three quantum states that is distributed.

We use the notation $|+--\rangle = |+\rangle\otimes|-\rangle\otimes|-\rangle$ etc., where the single spin states $|\pm\rangle$ are orthogonal states in a basis referred to as *basis I*. The three spinning particles are distributed to A, B and C, one particle to each of them, with the the first state in the tensor product corresponding to the spin sent to A, the second one to B and the third one to C. We assume the three-spin state is preserved under this distribution.

Each person in the receiving party can make (spin) measurements on the spinning particle he/she receives. The three can also communicate over a classical channel, which means that they can correlate their measurements and also compare the results of the measurements. They have, however no quantum channel available for communication. This means that all the observables that are available for measurements by the receiving party are of product form.

a) Determine the reduced density operator of A, and explain why, for any measurement he/she performs on his particle, no information can be extracted about which of the three spin states $|\psi_n\rangle$ is distributed. Also show that if A, B and C all make their spin measurements in *basis I*, even if they communicate their measured results, these cannot make any distinction between the three values of n.

Next, consider the situation where A and B are not able to communicate with C. They decide to perform measurements on the two spins they have received, and to make a probabilistic evaluation for the different values of n, based on the measured results. In order to do so they decide both to make their spin measurements in a rotated basis, which we refer to as *basis II*. The vectors in this basis are

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{4}$$

The possible outcomes of the measurements they list with numbers $k = 1, 2, 3, 4$, with the correspondence

$$k = 1: \ (0\,,0)\,, \quad k = 2: \ (0\,,1)\,, \quad k = 3: \ (1\,,0)\,, \quad k = 4: \ (1\,,1) \tag{5}$$

We refer to the corresponding states as $|\phi_k\rangle$, with $|\phi_1\rangle = |00\rangle = |0\rangle \otimes |0\rangle$, etc.

Before they do the measurements they evaluate for each three-spin state $|\psi_n\rangle$ the probabilities for the different measurement results (labeled by k). These probabilities are referred to as $p(k|n)$.

b) Find the reduced density operator $\hat{\rho}_n^{AB}$ and determine the probabilities $p(k|n)$ for different values of k and n. It is sufficient, due to repetitions of results, to consider $n = 0, 1$ and $k = 1, 2$. Do you, in particular, see a reason why the probabilities are the same for $n = 1$ and $n = 2$, for all k?

c) Assume now that A and B perform their measurements, with the result labeled by k. The probability for the state to be $|\psi_n\rangle$, under the condition that the measured result is k, we denote by $\bar{p}(n|k)$. Under the assumption that all spin states $|\psi_n\rangle$ are equally probable until the result of the measurement is known, statistics theory gives us the following relation

$$\bar{p}(n|k) = \frac{p(k|n)}{p(k)} \tag{6}$$

with $p(k)$ as a normalization factor. Determine $p(k)$ and the probability $\bar{p}(n|k)$ for each n in the case $k = 1: \ (0\,,0)$. What message is in this case most probably the one that has been distributed?