Midterm Exam FYS4110/9110, 2015

Solutions

Problem 1

a) Spin compositions

spin ½ x spin ½ = spin 0 + spin 1

with spin 0 and spin 1 defining orthogonal subspaces in the composite Hilbert space

Repeated

spin ½ x (spin ½ x spin ½) = spin ½ x spin 0 + spin ½ x spin 1

= spin ½ + spin ½ + spin 3/2

defining three orthogonal subspaces in the full Hilbert space.

b) Scalar products

$$\langle \psi_n | \psi_{n'} \rangle = \frac{1}{3}\left(1 + e^{2\pi i (n'-n)/3} + e^{-2\pi i (n'-n)/3}\right)$$

$$= \frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi}{3}(n'-n)\right)\right)$$

$n' = n \Rightarrow \cos\left(\frac{2\pi}{3}(n'-n)\right) = \cos 0 = 1$

$n' = \pm n \Rightarrow \cos\left(\frac{2\pi}{3}(n'-n)\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$

$\Rightarrow \underline{\langle \psi_n | \psi_{n'} \rangle = \delta_{nn'}}$ orthogonal for $n \neq n'$

$$\hat{S}_z |\psi_n\rangle = \frac{\hbar}{2}(1-1-1)|\psi_n\rangle = \underline{-\frac{\hbar}{2}|\psi_n\rangle}$$

Use lowering operator in the spectrum of $\hat{S}_z$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y = \hat{S}_{-1} + \hat{S}_{-2} + \hat{S}_{-3}$$

For single spin $\hat{S}_-|u\rangle = |d\rangle$, $\hat{S}_-|d\rangle = 0$

For the three spins

$$\hat{S}_-|udd\rangle = \hat{S}_-|dud\rangle = \hat{S}_-|ddu\rangle = |ddd\rangle$$

$$\Rightarrow \hat{S}_-|\psi_n\rangle = \frac{1}{\sqrt{3}}(1 + e^{2\pi i n/3} + e^{-2\pi i n/3})|ddd\rangle$$

$$= \frac{1}{\sqrt{3}}(1 + 2\cos(\tfrac{2\pi n}{3}))|ddd\rangle$$

$$\cos(\pm\tfrac{2\pi}{3}) = -\tfrac{1}{2} \Rightarrow$$

$$\underline{\hat{S}_-|\psi_0\rangle = \sqrt{3}\,|ddd\rangle} \qquad \underline{\hat{S}_-|\psi_{\pm 1}\rangle = 0}$$

This shows that $|\psi_\pm\rangle$ have no component with $s = \frac{3}{2}$

$\Rightarrow$ they are $s = \frac{1}{2}$ states $(\vec{S}^2 = \frac{3}{4}\hbar^2)$

This implies that $|\psi_0\rangle$ is the $s = \frac{3}{2}$ state $(\vec{S}^2 = \frac{15}{4}\hbar^2)$

c) Reduced density operator of spin 1

$$\hat{\rho}_1 = \mathrm{Tr}_{23}\Big(\tfrac{1}{3}(|udd\rangle\langle udd| + |dud\rangle\langle dud| + |ddu\rangle\langle ddu|$$

$$+ e^{2\pi i n/3}(|dud\rangle\langle udd| + |udd\rangle\langle ddu|)$$

$$+ e^{-2\pi i n/3}(|udd\rangle\langle dud| + |ddu\rangle\langle udd|)$$

$$+ e^{4\pi i n/3}|dud\rangle\langle ddu| + e^{-4\pi i n/3}|ddu\rangle\langle dud|\Big)$$

$$= \tfrac{1}{3}|u\rangle\langle u| + \tfrac{2}{3}|d\rangle\langle d|$$

Entanglement entropy for the 1(23) bipartite system

$$S_1 = -\tfrac{1}{3}\log\tfrac{1}{3} - \tfrac{2}{3}\log\tfrac{2}{3} = \log 3 - \tfrac{2}{3}\log 2 = \underline{0.918}$$

max value $S_{1max} = \log 2 = \underline{1}$ (both $\log = \log_2$)

The entanglement entropy is the same for all n, close to but somewhat smaller than the max. value

The symmetry with respect to permuting the spins implies that the other partitions give the same value

d) Measurement of $\hat{S}_{1z}$

The state of spin 1 is projected to $|u\rangle$ or $|d\rangle$ depending on the result

A Result: spin up

$$|\psi_n\rangle \rightarrow |udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$$

product state : no entanglement

B Result: spin down

$$|\psi_n\rangle \rightarrow |d\rangle \otimes |\phi_n\rangle$$

$$|\phi_n\rangle = \frac{1}{\sqrt{2}}\left(e^{2\pi i n/3}|ud\rangle + e^{-2\pi i n/3}|du\rangle\right)$$

$$\hat{\rho}_n = |\phi_n\rangle\langle\phi_n| = \frac{1}{2}\left(|ud\rangle\langle ud| + |du\rangle\langle du| + \text{cross terms}\right)$$

Reduced density operators

$$\hat{\rho}_{n1} = \hat{\rho}_{n2} = \frac{1}{2}\left(|u\rangle\langle u| + |d\rangle\langle d|\right) = \frac{1}{2}\mathbb{1}$$

Spin 2 and 3 are now in a maximally mixed state

e) New state

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|uuu\rangle - |ddd\rangle\right)$$

Reduced density operator

$$\hat{\rho}_1 = \mathrm{Tr}_{23}(|\phi\rangle\langle\phi|) = \frac{1}{2}\mathrm{Tr}_{23}\left(|uuu\rangle\langle uuu| + |ddd\rangle\langle ddd| + \text{cross terms}\right)$$

$$= \frac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|$$

$$= \frac{1}{2}\mathbb{1}$$

Entanglement entropy of partition 1(23)

$$S_1 = \underline{\log 2} = 1 \quad \text{maximal entanglement}$$

The same for the other partitions due to the symmetry of $|\phi\rangle$ under permutation of the spins

f) $|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$, $|b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$

$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)$, $|\ell\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$

$$\Rightarrow \quad |u\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) = \frac{1}{\sqrt{2}}(|r\rangle + |\ell\rangle)$$

$$|d\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle) = -\frac{i}{\sqrt{2}}(|r\rangle - |\ell\rangle)$$

$$\Rightarrow \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle)$$

$$= \frac{1}{2}(|bbb\rangle + |f^2b\rangle + |fbf\rangle + |bff\rangle)$$

$$= \underline{\frac{1}{2}(|rrf\rangle + |\ell\ell f\rangle + |r\ell b\rangle + |\ell r b\rangle)}$$

Measurement of S_{2z} <u>or</u> S_{3z} determines S_{1z}

Measurement of S_{2x} and S_{3x}:

outcomes $(bb)_{23} \Rightarrow b_1$
$(fb)_{23} \Rightarrow f_1$
$(bf)_{23} \Rightarrow f_1$
$(ff)_{23} \Rightarrow b_1$

$\Bigg\}$ determines <u>uniquely S_{x1}</u>

Measurement of S_{y2} and S_{3x}

outcomes: $(rf)_{23} \Rightarrow r_1$
$(\ell f)_{23} \Rightarrow \ell_1$
$(\ell b)_{23} \Rightarrow r_1$
$(rb)_{23} \Rightarrow \ell_1$

$\Bigg\}$ determines <u>uniquely S_{y1}</u>

Problem 2

a) Total spin $\vec{S} = \frac{\hbar}{2}(\vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{\sigma}) \equiv \frac{\hbar}{2}(\vec{\Sigma}_A + \vec{\Sigma}_B)$

$$\vec{S}^2 = \frac{\hbar^2}{2}(3\,\mathbb{1}\otimes\mathbb{1} + \vec{\Sigma}_A \cdot \vec{\Sigma}_B)$$

$$= \frac{\hbar^2}{2}\left(3\mathbb{1} + \sum_{k=1}^{3} \sigma_k \otimes \sigma_k\right)$$

$$\sigma_k \otimes \sigma_k |\psi_a\rangle = -|\psi_a\rangle$$

$$\sigma_z \otimes \sigma_z |\psi_s\rangle = -|\psi_s\rangle$$

$$\sigma_x \otimes \sigma_x |\psi_s\rangle = \sigma_y \otimes \sigma_y |\psi_s\rangle = |\psi_s\rangle$$

The three cases

I $\quad \langle \vec{S}^2 \rangle_1 = \frac{\hbar^2}{2}(3-3) = \underline{0}$

II $\quad \langle \vec{S}^2 \rangle_2 = \frac{\hbar^2}{2}(3+1) = \underline{2\hbar^2}$

III $\quad \langle \vec{S}^2 \rangle_3 = \frac{1}{2}(\langle \vec{S}^2 \rangle_1 + \langle \vec{S}^2 \rangle_2) = \underline{\hbar^2}$

$\hat{\rho}_1$ is a spin 0 state, $\hat{\rho}_2$ is a spin 1 state and $\hat{\rho}_3$ is a mixed state composed of spin 0 and 1

$\Rightarrow$ Only $\hat{\rho}_1$ is rotationally invariant

b) Reduced density operators

$$\hat{\rho}_1^A = \mathrm{Tr}_B\Big[\tfrac{1}{2}\big(|+-\rangle\langle+-| + |-+\rangle\langle-+| \underbrace{- |+-\rangle\langle-+| - |-+\rangle\langle+-|}_{\text{cross terms}}\big)\Big]$$

$$= \tfrac{1}{2}\big(|+\rangle\langle+| + |-\rangle\langle-|\big)_A = \underline{\tfrac{1}{2}\mathbb{1}_A}$$

Since the cross terms do not contribute:

$$\left.\begin{aligned} &\hat{\rho}_2^A = \hat{\rho}_3 = \hat{\rho}_1 = \underline{\tfrac{1}{2}\mathbb{1}_A} \\ &\text{Similarly } \hat{\rho}_1^B = \hat{\rho}_2^B = \hat{\rho}_3^B = \underline{\tfrac{1}{2}\mathbb{1}_B} \end{aligned}\right\} \text{maximally mixed}$$

$\hat{\rho}_1$ and $\hat{\rho}_2$ are pure states $\Rightarrow$ entropies $\underline{S_1 = S_2 = 0}$

$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)$ is mixed with probabilities $p_1 = p_2 = \frac{1}{2}$

$\Rightarrow$ entropy $S_3 = -p_1 \log p_1 - p_2 \log p_2 = \underline{\log 2}$

Entropies of subsystems

$S_1^A = S_2^A = S_3^A = \underline{\log 2}$, same for B

Inequality: $S_{max} \geq \max\{S_A, S_B\}$

I and II: not satisfied

III: satisfied as equality

Degree of entanglement

I and II are pure states,

entanglement entropies $\underline{S_1^A = S_2^A = \log 2}$, same for B

maximally entangled

$\underline{\text{III}}$: $\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) = \frac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+|)$

$= \frac{1}{2}(|+\rangle\langle+| \otimes |-\rangle\langle-| + |-\rangle\langle-| \otimes |+\rangle\langle+|)$

mixture of product states $\Rightarrow$ separable

no entanglement

c) $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \Rightarrow$

$\hat{S}_\theta|\theta\rangle = (\cos\theta\, S_z + \sin\theta\, S_x)(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle)$

$= \frac{\hbar}{2}\left[(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2})|+\rangle + (\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2})|-\rangle\right]$

$\underline{= \frac{\hbar}{2}(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle) = |\theta\rangle}$

$$P_A = Tr_A(\hat{\rho}_A \hat{P}(\theta)) = \langle\theta| \tfrac{1}{2}\mathbb{1}_A |\theta\rangle = \underline{\tfrac{1}{2}}$$

This is valid for all three cases I, II and III, it means that the probabilities for spin up and down are equal for any direction θ.

d) Joint probabilities

$$P(\theta,\theta') = Tr\,(\hat{\rho}\ \hat{P}(\theta)\otimes\hat{P}(\theta'))$$
$$= \langle\theta,\theta'|\hat{\rho}|\theta,\theta'\rangle \qquad |\theta;\theta'\rangle = |\theta\rangle\otimes|\theta'\rangle$$

$$\langle+-|\theta,\theta'\rangle = \langle+|\theta\rangle\langle-|\theta'\rangle = \cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}$$
$$\langle-+|\theta,\theta'\rangle = \langle-|\theta\rangle\langle+|\theta'\rangle = \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}$$

Case I :

$$P_1(\theta,\theta') = \tfrac{1}{2}[\langle\theta\theta'|+-\rangle\langle+-|\theta\theta'\rangle + \langle\theta\theta'|-+\rangle\langle-+|\theta\theta'\rangle$$
$$-\langle\theta\theta'|+-\rangle\langle-+|\theta\theta'\rangle - \langle\theta\theta'|-+\rangle\langle+-|\theta\theta'\rangle]$$
$$= \tfrac{1}{2}[\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta'}{2} + \sin^2\tfrac{\theta}{2}\cos^2\tfrac{\theta'}{2} - 2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\theta'}{2}]$$
$$= \tfrac{1}{2}(\cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2})^2$$
$$= \underline{\tfrac{1}{2}\sin^2\tfrac{\theta-\theta'}{2}}$$

Similar evaluations for case II and III

$$\underline{P_2(\theta,\theta') = \tfrac{1}{2}\sin^2\tfrac{\theta+\theta'}{2}}\ ,\quad \underline{P_3(\theta,\theta') = \tfrac{1}{4}(\sin^2\tfrac{\theta-\theta'}{2} + \sin^2\tfrac{\theta+\theta'}{2})}$$

f) Experimental quantities

$$P^A_{exp}(\theta) = \underline{\frac{n_{++}+n_{+-}}{N}}\ ,\quad P^B_{exp}(\theta) = \underline{\frac{n_{++}+n_{-+}}{N}}$$

$$P_{exp}(\theta,\theta') = \underline{\frac{n_{++}}{N}}$$

e) Plots of the function $F(\theta, \theta')$

Left: Plot of the curves $F(\theta, \theta/2)$ for cases I, II, III

Right: 3D plots of $F(\theta, \theta')$

Cases I and II: Bell's inequality broken (negative F, colored red in 3D plot

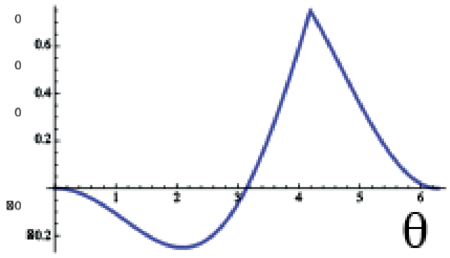
Case III: Bell's inequality unbroken (F positive)

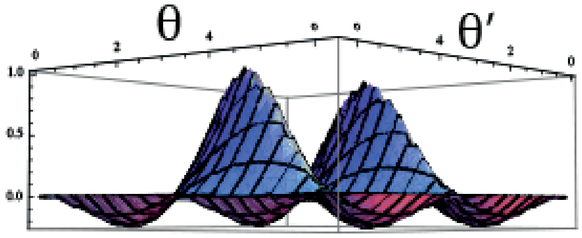
Results consistent with b): I an II entangled state, III non-entangled

$\theta'=0.5\,\theta$

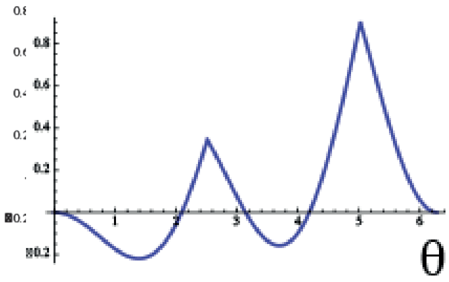
3D plot

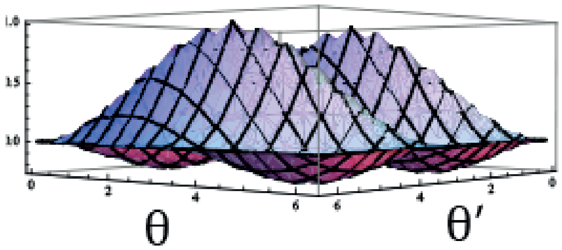
Case I





Case II





Case III