FYS4110/9110, Exam 2011

Solutions

Problem 1

a) Matrix elements of the Hamiltonian

$$\hat{H}\,|-,1\rangle = (-\tfrac{1}{2}\hbar\omega_0 + \hbar\omega)\,|-,1\rangle - i\hbar\lambda\,|+,0\rangle$$

$$\hat{H}\,|+,0\rangle = \tfrac{1}{2}\hbar\omega_0\,|+,0\rangle + i\hbar\lambda\,|-,1\rangle$$

$$\Rightarrow \langle -,1|\hat{H}|-,1\rangle = \tfrac{1}{2}\hbar(2\omega-\omega_0)$$

$$\langle +,0|\hat{H}|+,0\rangle = \tfrac{1}{2}\hbar\omega_0$$

$$\langle -,1|\hat{H}|+,0\rangle = i\hbar\lambda$$

$$\langle +,0|\hat{H}|-,1\rangle = -i\hbar\lambda$$

in matrix form

$$H = \tfrac{1}{2}\hbar\begin{pmatrix}\omega_0 & -2i\lambda\\ 2i\lambda & 2\omega-\omega_0\end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix}\omega_0-\omega & -2i\lambda\\ 2i\lambda & \omega-\omega_0\end{pmatrix} + \tfrac{1}{2}\hbar\omega\,\mathbb{1}$$

$$= \tfrac{1}{2}\hbar\Delta\begin{pmatrix}\cos\varphi & -i\sin\varphi\\ i\sin\varphi & -\cos\varphi\end{pmatrix} + \varepsilon\mathbb{1}$$

with $\Delta\cos\varphi = \omega_0-\omega$, $\Delta\sin\varphi = 2\lambda$, $\underline{\varepsilon = \tfrac{1}{2}\hbar\omega}$

$$\Rightarrow \underline{\Delta = \sqrt{(\omega-\omega_0)^2+4\lambda^2}},\quad \underline{\cos\varphi = \frac{\omega_0-\omega}{\Delta}},\quad \underline{\sin\varphi = \frac{2\lambda}{\Delta}}$$

b) Eigenvectors determined by

$$\begin{pmatrix}\cos\varphi & -i\sin\varphi\\ i\sin\varphi & -\cos\varphi\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mu\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{vmatrix}\cos\varphi-\mu & -i\sin\varphi\\ i\sin\varphi & -\cos\varphi-\mu\end{vmatrix} = 0 \quad\Rightarrow\quad \mu = \pm 1$$

Energies $E_\pm = \tfrac{1}{2}\hbar\omega \pm \tfrac{1}{2}\hbar\Delta = \underline{\tfrac{1}{2}\hbar\left(\omega \pm \sqrt{(\omega-\omega_0)^2+4\lambda^2}\right)}$

Eigenvectors

$$\cos\varphi\, \alpha_\pm - i\sin\varphi\, \beta_\pm = \pm \alpha_\pm$$

$$(\cos\varphi \mp 1)\,\alpha_\pm - i\sin\varphi\, \beta_\pm = 0$$

$$\Rightarrow \quad \alpha_\pm = N i\sin\varphi\,, \quad \beta_\pm = N(\cos\varphi \mp 1)$$

normalization $N^2(\sin^2\varphi + (\cos\varphi \mp 1)^2) = 1$

$$\Rightarrow \quad N = \frac{1}{\sqrt{2(1\mp\cos\varphi)}}$$

$$\psi_\pm(\varphi) = \frac{1}{\sqrt{2(1\mp\cos\varphi)}} \begin{pmatrix} i\sin\varphi \\ \cos\varphi \mp 1 \end{pmatrix}$$

$$\sin\varphi = 2\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\,; \quad \cos\varphi = 2\cos^2\tfrac{\varphi}{2} - 1 = 1 - 2\sin^2\tfrac{\varphi}{2}$$

$$\Rightarrow \quad |\psi_+(\varphi)\rangle = -\sin\tfrac{\varphi}{2}\,|-,1\rangle + i\cos\tfrac{\varphi}{2}\,|+,0\rangle$$

$$|\psi_-(\varphi)\rangle = \cos\tfrac{\varphi}{2}\,|-,1\rangle + i\sin\tfrac{\varphi}{2}\,|+,0\rangle$$

$$\cos\left(\tfrac{\varphi+\pi}{2}\right) = -\sin\tfrac{\varphi}{2}\,, \quad \sin\left(\tfrac{\varphi+\pi}{2}\right) = \cos\tfrac{\varphi}{2}$$

$$\Rightarrow \quad |\psi_-(\varphi+\pi)\rangle = |\psi_+(\varphi)\rangle$$

c) Density operator of the $|\psi_-(\varphi)\rangle$ state

$$\rho(\varphi) = |\psi_-(\varphi)\rangle\langle\psi_-(\varphi)|$$

$$= \cos^2\tfrac{\varphi}{2}\,|-,1\rangle\langle -,1| + \sin^2\tfrac{\varphi}{2}\,|+,0\rangle\langle +,0| + i\cos\tfrac{\varphi}{2}\sin\tfrac{\varphi}{2}\left(|+,0\rangle\langle -,1| - |-,1\rangle\langle +,0|\right)$$

$$\rho_{ph}(\varphi) = \langle -|\rho(\varphi)|-\rangle + \langle +|\rho(\varphi)|+\rangle = \sin^2\tfrac{\varphi}{2}\,|0\rangle\langle 0| + \cos^2\tfrac{\varphi}{2}\,|1\rangle\langle 1|$$

$$\rho_{atom}(\varphi) = \langle 0|\rho(\varphi)|0\rangle + \langle 1|\rho(\varphi)|1\rangle = \cos^2\tfrac{\varphi}{2}\,|-\rangle\langle -| + \sin^2\tfrac{\varphi}{2}\,|+\rangle\langle +|$$

$\cos^2\frac{\varphi}{2} > \sin^2\frac{\varphi}{2}$ $\left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right)$: the state is mainly a one-photon state

$\cos^2\frac{\varphi}{2} < \sin^2\frac{\varphi}{2}$ $\left(\frac{\pi}{2} < \varphi < 3\frac{\pi}{2}\right)$: the state is mainly an excited atomic state

d) Entanglement entropy

$$S = -\mathrm{Tr}_{ph}(\rho_{ph}\log\rho_{ph}) = -\mathrm{Tr}_{atom}(\rho_{atom}\log\rho_{atom})$$

$$= -\left(\cos^2\tfrac{\varphi}{2}\log(\cos^2\tfrac{\varphi}{2}) + \sin^2\tfrac{\varphi}{2}\log(\sin^2\tfrac{\varphi}{2})\right)$$

Min. value when $|\psi_-(\varphi)\rangle$ is a product state:

$\cos\frac{\varphi}{2} = 0$ or $\sin\frac{\varphi}{2} = 0 \Rightarrow \varphi = 0, \pi$

gives $\underline{S = 0}$

Max. value, when ρ_{ph} (ρ_{atom}) is maximally mixed:

$\cos^2\frac{\varphi}{2} = \sin^2\frac{\varphi}{2} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{2}, 3\frac{\pi}{2}$

$\Rightarrow \rho_{ph} = \frac{1}{2}\mathbb{1} \Rightarrow \underline{S = \log 2}$ max. entangled

e) Time evolution: expand in energy eigenstates

$$|\psi(0)\rangle = |-,1\rangle = \cos\frac{\varphi}{2}|\psi_-(\varphi)\rangle - \sin\frac{\varphi}{2}|\psi_+(\varphi)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \cos\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_-t}|\psi_-(\varphi)\rangle - \sin\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_+t}|\psi_+(\varphi)\rangle$$

$$= \left(\cos^2\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_-t} + \sin^2\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_+t}\right)|-,1\rangle$$

$$+ i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\left(e^{-\frac{i}{\hbar}E_-t} - e^{-\frac{i}{\hbar}E_+t}\right)|+,0\rangle$$

Probability for a photon present

$$p(t) = |\langle -,1|\psi(t)\rangle|^2 = \cos^4\frac{\varphi}{2} + \sin^4\frac{\varphi}{2}$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\left(e^{-\frac{i}{\hbar}(E_--E_+)t} + e^{+\frac{i}{\hbar}(E_--E_+)t}\right)$$

$$= \frac{1}{4}(1+\cos\varphi)^2 + \frac{1}{4}(1-\cos\varphi)^2 + \frac{1}{2}\sin^2\varphi\cos\left(\frac{E_--E_+}{\hbar}t\right)$$

$$= \underline{\frac{1}{2}\left(1+\cos^2\varphi + \sin^2\varphi\cos\Delta t\right)} \qquad \Delta = \sqrt{(\omega-\omega_0)^2 + 4\lambda^2}$$

Oscillations due to time dependent mixing of the one-photon state with the excited atom state. Frequency $\underline{\Delta}$,

amplitude $\frac{1}{2}\sin^2\varphi = \underline{\frac{2\lambda^2}{(\omega-\omega_0)^2 + 4\lambda^2}}$

Problem 2

a) Time evolution of the two-level system, $\kappa = 0$:

$$\mathcal{U}(t) = e^{-\frac{i}{2}\omega_A t \sigma_z} = \begin{pmatrix} e^{-\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{\frac{i}{2}\omega_A t} \end{pmatrix}$$

$$\rho_A(t) = \mathcal{U}(t)\rho_A(0)\mathcal{U}^\dagger(t)$$

$$= \begin{pmatrix} e^{-\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{\frac{i}{2}\omega_A t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{-\frac{i}{2}\omega_A t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1+z & e^{-i\omega_A t}(x-iy) \\ e^{i\omega_A t}(x+iy) & 1-z \end{pmatrix} \Rightarrow x(t) + iy(t) = e^{i\omega_A t}(x+iy)$$

$$\Rightarrow x(t) = x\cos\omega_A t - y\sin\omega_A t$$
$$y(t) = x\sin\omega_A t + y\cos\omega_A t$$
$$z(t) = z$$

Precession of $\vec{r}$ around the z-axis, with ang. freq. ω_A

b) Interaction matrix element

$$\langle -, 1_k | \hat{H}_{int} | +, 0 \rangle = \kappa \sqrt{\frac{\hbar}{2L\omega_k}}$$

decay rate:

$$\gamma = \frac{L}{(2\pi\hbar)^2} \int dk \, \frac{\kappa\hbar^2}{2L\omega_k} \, \delta(\omega_k - \omega_A) \qquad k = \frac{\omega_k}{c}$$

$$= \frac{L}{4\pi^2\hbar^2} \frac{\kappa^2\hbar}{2Lc\omega_A} = \underline{\frac{\kappa^2}{8\pi^2\hbar c\,\omega_A}}$$

c) $|\psi(t)\rangle = |\phi(t)\rangle \otimes |0\rangle + \sum_k c_k(t) |-, 1_k\rangle$

with $|\phi(t)\rangle = e^{-\frac{i}{2}\omega_A t - \gamma t/2} \alpha |+\rangle + e^{\frac{i}{2}\omega_A t} \beta |-\rangle$

Normalization

$$\langle\psi(t)|\psi(t)\rangle = \langle\phi(t)|\phi(t)\rangle + \sum_k |c_k(t)|^2$$

$$= e^{-\gamma t} |\alpha|^2 + |\beta|^2 + \sum_k |c_k(t)|^2 \overset{!}{=} 1$$

$$\Rightarrow \sum_k |c_k(t)|^2 = \underline{|\alpha|^2 (1 - e^{-\gamma t})}$$

Reduced density operator of the two-level system

$$\rho_A(t) = Tr_B(|\psi(t)\rangle\langle\psi(t)|) = |\phi(t)\rangle\langle\phi(t)| + \sum_k |c_k(t)|^2 |-\rangle\langle-|$$

$$\underline{= e^{-\gamma t} |\alpha|^2 |+\rangle\langle+| + (1 - e^{-\gamma t}|\alpha|^2) |-\rangle\langle-| + e^{-\gamma t/2} (\alpha\beta^* e^{-i\omega_A t} |+\rangle\langle-| + \alpha^*\beta e^{i\omega_A t} |-\rangle\langle+|)}$$

d) $\alpha = 1$, $\beta = 0$:

$$\rho_A(t) = e^{-\gamma t} |+\rangle\langle+| + (1 - e^{-\gamma t}) |-\rangle\langle-|$$

$$= \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix}$$

$$\Rightarrow z(t) = \underline{2 e^{-\gamma t} - 1}, \quad \underline{x(t) = y(t) = 0}$$

The excited state decays exponentially into the ground state, as expected

$t = 0$ and $t \to \infty$ ($z = \pm 1$) pure product state, $S_A = 0$

Intermediate time: $e^{-\gamma t} = \frac{1}{2} \Rightarrow \rho_A = \frac{1}{2}\mathbb{1}$, maximally entangled

e) $\alpha = \beta = \frac{1}{\sqrt{2}}$:

$$\rho_A(t) = \tfrac{1}{2} e^{-\gamma t} |+\rangle\langle+| + (1 - \tfrac{1}{2} e^{-\gamma t}) |-\rangle\langle-|$$
$$+ \tfrac{1}{2} e^{-\gamma t/2} \left(e^{-i\omega_A t} |+\rangle\langle-| + e^{i\omega_A t} |-\rangle\langle+| \right)$$
$$= \frac{1}{2} \begin{pmatrix} e^{-\gamma t} & e^{-\gamma t/2} e^{-i\omega_A t} \\ e^{-\gamma t/2} e^{i\omega_A t} & 2 - e^{-\gamma t} \end{pmatrix} \Rightarrow x(t) + i y(t) = e^{-\gamma t/2} e^{i\omega_A t}$$

$\underline{x(t) = e^{-\gamma t/2} \cos\omega_A t}$, $\underline{y(t) = e^{-\gamma t/2} \sin\omega_A t}$; $\underline{z(t) = e^{-\gamma t} - 1}$

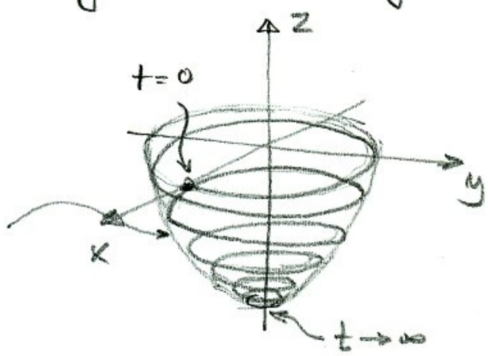
Combination of motions in a) and d) :

$\gamma \ll \omega_A \Rightarrow$ rapid precession of $\vec{r}$ around the z-axis, combined with slow decay towards the ground state

Sketch of the motion

$x^2 + y^2 = z + 1$

$\Rightarrow$ parabolic surface



$r^2 = e^{-\gamma t} + (e^{-\gamma t} + 1)^2$

$\underline{= 1 - e^{-\gamma t} + e^{-2\gamma t}}$

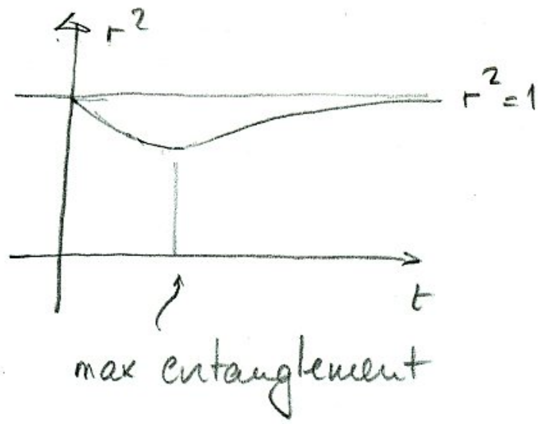
$t = 0: r^2 = 1$, $t \to \infty: r^2 \to 1$ ent. entropy $S_A = 0$

Intermediate times $0 < r^2 < 1$

min value for $e^{-\gamma t} = \frac{1}{2}$

$\Rightarrow r^2 = \frac{3}{4}$

gives max value for S_A



FYS4110 Eksamensoppgaver 2012

Løsninger

Problem 1

a) Hamiltonian applied to the product states

$$\hat{H}|++\rangle = \tfrac{1}{2}\hbar(\omega_1+\omega_2)|++\rangle$$

$$\hat{H}|--\rangle = -\tfrac{1}{2}\hbar(\omega_1+\omega_2)|--\rangle$$

$$\hat{H}|+-\rangle = \tfrac{1}{2}\hbar\Delta|+-\rangle + \tfrac{1}{2}\hbar\lambda|-+\rangle$$

$$\hat{H}|-+\rangle = -\tfrac{1}{2}\hbar\Delta|-+\rangle + \tfrac{1}{2}\hbar\lambda|+-\rangle$$

In the subspace spanned by $|+-\rangle$ and $|-+\rangle$,

$$H = \tfrac{1}{2}\hbar\begin{pmatrix}\Delta & \lambda \\ \lambda & -\Delta\end{pmatrix} = \tfrac{1}{2}\hbar\mu\begin{pmatrix}\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}$$

The matrix is determined by φ, with μ as a scale factor. This implies that the eigenstates are determined by φ.

b) Eigenvalues in subspace

$$\begin{vmatrix}\cos\varphi-\varepsilon & \sin\varphi \\ \sin\varphi & -\cos\varphi-\varepsilon\end{vmatrix} = 0 \Rightarrow \varepsilon_\pm = \pm 1 .$$

energies $E_\pm = \pm\tfrac{1}{2}\hbar\mu = \underline{\pm\tfrac{1}{2}\hbar\sqrt{\Delta^2+\lambda^2}}$

Eigenstates

$$(\cos\varphi \mp 1)\alpha_\pm + \sin\varphi\,\beta_\pm = 0$$

$$(\cos\varphi \pm 1)\beta_\pm - \sin\varphi\,\alpha_\pm = 0$$

$$\Rightarrow (\cos\varphi \mp 1)\beta_\mp - \sin\varphi\,\alpha_\mp = 0$$

$$\frac{\beta_+}{\alpha_+} = -\frac{\alpha_-}{\beta_-} = -\frac{\cos\varphi - 1}{\sin\varphi} = \frac{2\sin^2\varphi/2}{2\cos\varphi/2\,\sin\varphi/2} = \tan\varphi/2$$

Normalized solutions

$\alpha_+ = \cos\frac{\varphi}{2} \quad \beta_+ = \sin\frac{\varphi}{2}$ $\qquad |\psi_+\rangle = \cos\frac{\varphi}{2}|+-\rangle + \sin\frac{\varphi}{2}|-+\rangle$

$\alpha_- = \sin\frac{\varphi}{2} \quad \beta_- = -\cos\frac{\varphi}{2}$ $\qquad |\psi_-\rangle = \sin\frac{\varphi}{2}|+-\rangle - \cos\frac{\varphi}{2}|-+\rangle$

c) $\Delta = 0 \Rightarrow \cos\varphi = 0 \Rightarrow \varphi = \pi/2 \Rightarrow \cos\frac{\varphi}{2} = \sin\frac{\varphi}{2} = \frac{1}{\sqrt{2}}$

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle)$$

$$|\pm\mp\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle \pm |\psi_-\rangle) = |\psi(0)\rangle$$

Time evolution

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\mu t}|\psi_+\rangle + e^{\frac{i}{2}\mu t}|\psi_-\rangle\right) \qquad \mu = \lambda$$

$$= \frac{1}{2}\left(e^{-\frac{i}{2}\mu t}(|+-\rangle + |-+\rangle) + e^{\frac{i}{2}\mu t}(|+-\rangle - |-+\rangle)\right)$$

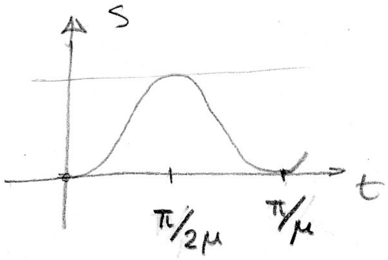
$$= \underline{\cos(\tfrac{\mu t}{2})|+-\rangle - i\sin(\tfrac{\mu t}{2})|-+\rangle} \equiv c(t)|+-\rangle + i s(t)|-+\rangle$$

Density operator

$$\rho(t) = c(t)^2|+-\rangle\langle+-| + s(t)^2|-+\rangle\langle-+|$$
$$+ c(t)s(t)\left(|+-\rangle\langle-+| + |-+\rangle\langle+-|\right)$$

Reduced density operators

$$\rho_1(t) = c(t)^2|+\rangle\langle+| + s(t)^2|-\rangle\langle-|$$
$$\rho_2(t) = c(t)^2|-\rangle\langle-| + s(t)^2|+\rangle\langle+|$$



Entanglement entropy

$$S_1 = S_2 = -c^2\log c^2 - s^2\log s^2$$

max value : $c^2 = s^2 = \frac{1}{2} \Rightarrow S_A = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = \log 2$

min value : $c^2 = 1 \vee s^2 = 1$ $\quad S = 0$ for $c = 0 \vee s = 0$, $t = 0, \frac{\pi}{\mu}, \frac{\pi}{2\mu}, \ldots$

period $\underline{T = \frac{\pi}{\mu}}$

Problem 2

a) Hamiltonian applied to the product states

$$\hat{H}|g,1\rangle = \hbar(\tfrac{1}{2}\omega - i\gamma)|g,1\rangle + \tfrac{1}{2}\hbar\lambda|e,0\rangle$$

$$\hat{H}|e,0\rangle = \tfrac{1}{2}\hbar\omega|e,0\rangle + \tfrac{1}{2}\hbar\lambda|g,1\rangle$$

$$\hat{H}|g,0\rangle = -\tfrac{1}{2}\hbar\omega|g,0\rangle$$

In the space spanned by $|g,1\rangle$ and $|e,0\rangle$

$$H = \tfrac{1}{2}\hbar(\omega - i\gamma)\mathbb{I} + \tfrac{1}{2}\hbar\begin{pmatrix} -i\gamma & \lambda \\ \lambda & i\gamma \end{pmatrix} \equiv H_0 + H_1$$

b) Define $|\psi(t)\rangle = e^{-\frac{i}{2}\omega t - \frac{1}{2}\gamma t}|\phi(t)\rangle$

$$|\phi(t)\rangle = (\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle + ib\sin(\Omega t)|g,1\rangle$$

$\Rightarrow \quad |\psi(0)\rangle = |\phi(0)\rangle = |e,0\rangle$

satisfies the initial condition

need to show that $|\psi(t)\rangle$ satisfies the Schrödinger eq.

Note $\quad i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \quad \mathrm{I}$

$\Leftrightarrow \quad i\hbar\frac{d}{dt}|\phi(t)\rangle = \hat{H}_1|\phi(t)\rangle \quad \mathrm{II}$

Need to show that II is satisfied

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = i\hbar\Omega\left[ib\cos(\Omega t)|g,1\rangle + (-\sin\Omega t + a\cos(\Omega t))|e,0\rangle\right]$$

$$\hat{H}_1|\phi(t)\rangle = \tfrac{1}{2}\hbar\{\gamma b\sin(\Omega t) + \lambda(\cos(\Omega t) + a\sin(\Omega t))\}|g,1\rangle$$

$$\tfrac{1}{2}\hbar(i\lambda b\sin\Omega t + i\gamma(\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle$$

$$= \tfrac{1}{2}\hbar\left[\{\lambda\cos(\Omega t) + (a\lambda + \gamma b)\sin(\Omega t)\}|g,1\rangle + i\{\gamma\cos\Omega t + (\lambda b + \gamma a)\sin(\Omega t)\}|e,0\rangle\right]$$

Conditions for equality

$$-\Omega b = \tfrac{1}{2}\lambda \quad \text{I}$$

$$a\lambda + \gamma b = 0 \quad \text{II}$$

$$\Omega a = \tfrac{1}{2}\gamma \quad \text{III}$$

$$-\Omega = \tfrac{1}{2}(\lambda b + \gamma a) \quad \text{IV}$$

$$\text{I} \Rightarrow \quad \underline{b = -\frac{\lambda}{2\Omega}} \qquad \text{III} \quad \underline{a = \frac{\gamma}{2\Omega}}$$

$$\Rightarrow a\lambda + \gamma b = \frac{\gamma\lambda - \gamma\lambda}{2\Omega} = 0$$ consistent with II

$$\text{IV} \Rightarrow \Omega = \frac{1}{4\Omega}(\lambda^2 - \gamma^2)$$

$$\Omega^2 = \frac{1}{4}(\lambda^2 - \gamma^2) \Rightarrow \underline{\Omega = \tfrac{1}{2}\sqrt{\gamma^2 - \lambda^2}}$$

c) Assume $\mathrm{Tr}\, \rho_{tot} = 1$

$$\Rightarrow \mathrm{Tr}\rho(t) + f(t) = 1 \qquad f(t) = 1 - \mathrm{Tr}\rho(t)$$

$$\mathrm{Tr}\rho(t) = \langle\psi(t)|\psi(t)\rangle = e^{-\gamma t}\langle\phi(t)|\phi(t)\rangle$$

$$\langle\phi(t)|\phi(t)\rangle = \cos^2(\Omega t) + a^2\sin^2(\Omega t) + 2a\cos\Omega t\sin\Omega t + b^2\sin^2\Omega t$$

$$= 1 + (a^2+b^2-1)\sin^2\Omega t + 2a\cos\Omega t\sin\Omega t$$

$$= 1 + \tfrac{1}{2}(a^2+b^2-1) - \tfrac{1}{2}(a^2+b^2-1)\cos(2\Omega t) + a\sin(2\Omega t)$$

$$a^2+b^2-1 = \frac{\lambda^2+\gamma^2}{\lambda^2-\gamma^2} - 1 = \frac{2\gamma^2}{\lambda^2-\gamma^2}$$

$$1 + \tfrac{1}{2}(a^2+b^2-1) = 1 + \frac{\gamma^2}{\lambda^2-\gamma^2} = \frac{\lambda^2}{\lambda^2-\gamma^2}$$

$$= \mathrm{Tr}\rho = \underline{e^{-\gamma t}\left(\frac{\lambda^2}{\lambda^2-\gamma^2} - \frac{\gamma^2}{\lambda^2-\gamma^2}\cos(\sqrt{\lambda^2-\gamma^2}\, t) + \frac{\gamma}{\sqrt{\lambda^2-\gamma^2}}\sin(\sqrt{\lambda^2-\gamma^2}\, t)\right)}$$

$$\underline{f(t) = 1 - \mathrm{Tr}\rho(t)}$$

The decay of $\mathrm{Tr}\rho$ is due to the leakage of the cavity photon out of the system. For the cavity states this corresponds to the transition $|g,1\rangle \to |g,0\rangle$. The second term in Eq. (5) gives the build up of probability in $|g,0\rangle$ due to this process.

With $\gamma = 0$, there are oscillations between $|g,1\rangle$ and $|e,0\rangle$ due to the coupling between the atom and the cavity photon. The time evolution of the state (4) shows, for $\gamma \neq 0$, decay of the probabilities due to the leakage $|g,1\rangle \to |g,0\rangle$, superimposed on these oscillations.

Problem 3

a) The full density operator

$$\rho_n = \tfrac{1}{3}\{ |+--\rangle\langle+--| + |-+-\rangle\langle-+-| + |--+\rangle\langle--+|$$
$$+ \eta^n (|-+-\rangle\langle+--| + |+--\rangle\langle--+|) + (\eta^*)^n (|+--\rangle\langle-+-| + |--+\rangle\langle+--|)$$
$$+ \eta^{2n} |-+-\rangle\langle--+| + (\eta^*)^{2n} |--+\rangle\langle-+-|$$

Reduced density operator

$$\rho_n^A = \mathrm{Tr}_{BC}\, \rho_n = \tfrac{1}{3}(|+\rangle\langle+| + 2|-\rangle\langle-|)$$ independent of n, information about n can therefore not be detected by A

Measurement by A, B, C in basis I, gives result determined by probabilities of the form $\langle abc|\rho_n|abc\rangle$ with $|abc\rangle$ as a product of states $|\pm\rangle$. Only the diagonal terms in ρ_n give contributions, and these are independent of n. Again there are no measurable differences between different n.

b) Reduced density operator

$$\rho_n^{AB} = \mathrm{Tr}_C \rho_n = \frac{1}{3}\{ |+-\rangle\langle +-| + |-+\rangle\langle -+| + |--\rangle\langle --|$$
$$+ \eta^n |-+\rangle\langle +-| + (\eta^*)^n |+-\rangle\langle -+| \}$$

probabilities $p(k|n) = \langle \phi_k | \rho_n^{AB} | \phi_k \rangle$

Need overlap between vectors of basis I and II:

$$\langle 0|+\rangle = \langle 0|-\rangle = \langle 1|+\rangle = \frac{1}{\sqrt{2}} \quad \langle 1|-\rangle = -\frac{1}{\sqrt{2}}$$

note: only sign change for $\langle 1|-\rangle$

$$p(1|0) = \langle 00|\rho_0^{AB}|00\rangle = \frac{1}{3}\,\frac{5}{4} = \frac{5}{12}$$
$$p(2|0) = \langle 01|\rho_0^{AB}|01\rangle = \frac{1}{3}\left(\frac{3}{4} - \frac{2}{4}\right) = \frac{1}{12}$$
$$p(1|1) = \langle 00|\rho_1^{AB}|00\rangle = \frac{1}{3}\left(\frac{3}{4} + \frac{\eta+\eta^*}{4}\right) = \frac{1}{6}$$
$$p(2|1) = \langle 01|\rho_1^{AB}|01\rangle = \frac{1}{3}\left(\frac{3}{4} - \frac{\eta+\eta^*}{4}\right) = \frac{1}{3}$$

Have used $\eta + \eta^* = -1$

The change $n=1 \to n=2$ corresponds to $\eta \to \eta^*$ since $\eta^2 = \eta^*$

no change since the probabilities are real

c) Normalization of probabilities

$$\sum_n \bar{p}(n|k) = 1 \Rightarrow p(k) = \sum_n p(k|n)$$

$$p(1) = p(1|0) + p(1|1) + p(1|2) = \frac{5}{12} + 2\cdot\frac{1}{6} = \frac{9}{12} = \frac{3}{4}$$

Probabilities for $k=1$, $n=0,1,2$

$$\bar{p}(0|1) = \frac{p(1|0)}{p(1)} = \frac{5}{12}\cdot\frac{12}{9} = \frac{5}{9}$$
$$\bar{p}(1|1) = \frac{p(1|1)}{p(1)} = \frac{1}{6}\cdot\frac{12}{9} = \frac{2}{9}$$
$$\bar{p}(2|1) = \text{— " —} = \frac{2}{9}$$

The message $n=0$ is most probable, with probability $\frac{5}{9}$,
while $n=1$ and 2 have probability $\frac{2}{9}$.

FYS4110/9110 Eksamen 2013

Løsninger

Oppgave 1

a) Utnytter $\hat{\alpha}^\dagger\hat{\alpha} = |e\rangle\langle g|g\rangle\langle e| = |e\rangle\langle e|$

Lindbladligning

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] - \frac{1}{2}\gamma\{|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e| - 2|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|\}$$

for matriseelementer, utnytter

$$\langle e|[\hat{H}_0, \hat{\rho}]|e\rangle = \langle g|[\hat{H}_0, \hat{\rho}]|g\rangle = 0$$

$$\langle e|[\hat{H}_0, \hat{\rho}]|g\rangle = (E_e - E_g)\langle e|\hat{\rho}|g\rangle = \hbar\omega\langle e|\hat{\rho}|g\rangle$$

$$\Rightarrow \quad \frac{dp_e}{dt} = -\gamma p_e \qquad p_e(t) = e^{-\gamma t}p_e(0)$$

$$\frac{dp_g}{dt} = \gamma p_e \Rightarrow p_g(t) = 1 - p_e(t)$$

$$\frac{db}{dt} = (-i\omega - \tfrac{1}{2}\gamma)\, b \Rightarrow b(t) = e^{-i\omega t - \frac{1}{2}\gamma t}\, b(0)$$

Initialbetingelser

$$p_e(0) = 1, \quad p_g(0) = 0, \quad b(0) = 0$$

$$\Rightarrow \quad \underline{p_e(t) = e^{-\gamma t}}, \quad \underline{p_g(t) = 1 - e^{-\gamma t}}, \quad \underline{b(t) = 0}$$

b) Nye initialbetingelser

$$p_e(0) = |\langle e|\psi\rangle|^2 = \tfrac{1}{2}$$

$$p_g(0) = |\langle g|\psi\rangle|^2 = \tfrac{1}{2}$$

$$b(0) = \langle e|\psi\rangle\langle\psi|g\rangle = \tfrac{1}{2}$$

Tidsutvikling

$p_e(t) = \frac{1}{2} e^{-\gamma t}$, $p_g(t) = 1 - \frac{1}{2} e^{-\gamma t}$, $b(t) = \frac{1}{2} e^{-i\omega t - \frac{1}{2}\gamma t}$

$$\Rightarrow \qquad \underline{\rho(t) = \frac{1}{2} \begin{pmatrix} e^{-\gamma t} & e^{-i\omega t - \frac{1}{2}\gamma t} \\ e^{i\omega t - \frac{1}{2}\gamma t} & 2 - e^{-\gamma t} \end{pmatrix}}$$

c)

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{r}.\vec{\sigma}) = \frac{1}{2} \begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$\Rightarrow z = p_e - p_g$, $x = 2\,\mathrm{Re}\,b$, $y = -2\,\mathrm{Im}\,b$

$\Rightarrow r^2 = (p_e - p_g)^2 + 4|b|^2$

Tilfelle a):

$r^2 = (2e^{-\gamma t} - 1)^2$

minimum for $e^{-\gamma t} = \frac{1}{2}$, $\underline{t = \frac{1}{\gamma} \ln 2}$ $\quad \underline{r_{min} = 0}$

$\Rightarrow \hat{\rho} = \frac{1}{2}\mathbb{1}$, maksimalt blandet $\Rightarrow$ A+B er maksimalt sammenfiltret.

Tilfelle b)

$r^2 = (e^{-\gamma t} - 1)^2 + e^{-\gamma t} = e^{-2\gamma t} - e^{-\gamma t} + 1$

$\frac{d}{dt} r^2 = 0 \Rightarrow -2e^{-2\gamma t} + e^{-\gamma t} = 0 \Rightarrow e^{-\gamma t} = \frac{1}{2}$, $\underline{t = \frac{1}{\gamma} \ln 2}$

$\Rightarrow r^2_{min} = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$, $\quad \underline{r_{min} = \frac{1}{2}\sqrt{3}}$

Siden $r_{min} < 1$ er $\hat{\rho}$ en blandet tilstand,

$\Rightarrow$ A+B er sammenfiltret, men mindre enn i tilfellet a)

I begge tilfeller er $r = 1$ både for $t = 0$ og $t \to \infty$, dvs. sammenfiltringen er bare midlertidig under henfallet $|\psi\rangle_{init} \to |g\rangle$.

Oppgave 2

a) Reduserte tetthetsoperatorer

$\hat{\rho}_A = \mathrm{Tr}_{BC}(|\psi\rangle\langle\psi|) = \frac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|) = \underline{\frac{1}{2}\mathbb{1}_A}$

$\hat{\rho}_{BC} = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \underline{\frac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd|)}$

$\hat{\rho}_A$ er maksimalt blandet $\Rightarrow$ sammenfiltringsentropien er maksimal: $S = -\mathrm{Tr}_A(\hat{\rho}_A \log \rho_A) = \log 2$

$\hat{\rho}_{BC}$ er separabel, dvs en sum av produkttilstander, $|u\rangle \otimes |u\rangle$ og $|d\rangle \otimes |d\rangle$. Ingen sammenfiltring

b) Uttrykker A-tilstanden i $|\frac{\pi}{2},+\rangle \equiv |f\rangle$ og $|\frac{\pi}{2},-\rangle \equiv |b\rangle$

$|u\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle)$, $|d\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle)$

$\Rightarrow |\psi\rangle = \frac{1}{2}|f\rangle \otimes (|uu\rangle + |dd\rangle) + \frac{1}{2}|b\rangle \otimes (|uu\rangle - |dd\rangle)$

Målingen gir f (spinn opp) $\Rightarrow$

$|\psi\rangle \to |\psi'\rangle = \frac{1}{\sqrt{2}}|f\rangle \otimes (|uu\rangle + |dd\rangle)$ normert

$\hat{\rho}_{BC} \to \hat{\rho}'_{BC} = \frac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd| + |uu\rangle\langle dd| + |dd\rangle\langle uu|)$

Dette er en ren tilstand

$\hat{\rho}_B = \mathrm{Tr}_C\, \hat{\rho}_{BC} = \frac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|) = \frac{1}{2}\mathbb{1}_B$

Denne er maksimalt blandet $\Rightarrow$ B+C er maks. sammenfiltret

Målingen på A gjør B+C sammenfiltret!

c) Roterte tilstander

$$|u\rangle = \cos\tfrac{\theta}{2}\,|\theta,+\rangle - \sin\tfrac{\theta}{2}\,|\theta,-\rangle$$

$$|d\rangle = \sin\tfrac{\theta}{2}\,|\theta,+\rangle + \cos\tfrac{\theta}{2}\,|\theta,-\rangle$$

$\Longrightarrow$

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\Big\{|\theta,+\rangle \otimes \big(\cos\tfrac{\theta}{2}\,|uu\rangle + \sin\tfrac{\theta}{2}\,|dd\rangle\big)$$

$$+\ |\theta,-\rangle \otimes \big(-\sin\tfrac{\theta}{2}\,|uu\rangle + \cos\tfrac{\theta}{2}\,|dd\rangle\big)\Big\}$$

Måleresultat $(\theta,+) \Rightarrow$

$$|\psi\rangle \to |\theta,+\rangle \otimes \big(\cos\tfrac{\theta}{2}\,|uu\rangle + \sin\tfrac{\theta}{2}\,|dd\rangle\big)$$

$$\equiv |\theta,+\rangle \otimes |\psi'_{BC}(\theta)\rangle$$

$\hat{\rho}_{BC} \to \hat{\rho}'_{BC} = |\psi'_{BC}\rangle\langle\psi'_{BC}|$ ren tilstand

$$= \cos^2\tfrac{\theta}{2}\,|uu\rangle\langle uu| + \sin^2\tfrac{\theta}{2}\,|dd\rangle\langle dd|$$

$$\underline{+ \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\,\big(|uu\rangle\langle dd| + |dd\rangle\langle uu|\big)}$$

Redusert tetthetsoperator

$$\hat{\rho}_B = \mathrm{Tr}_c\,\hat{\rho}_{BC} = \cos^2\tfrac{\theta}{2}\,|u\rangle\langle u| + \sin^2\tfrac{\theta}{2}\,|d\rangle\langle d|$$

$\langle u|d\rangle = 0 \Rightarrow \cos^2\frac{\theta}{2}$ og $\sin^2\frac{\theta}{2}$ er egenverdier til $\hat{\rho}_B$

Entropi $S = \underline{-\cos^2\tfrac{\theta}{2}\ln(\cos^2\tfrac{\theta}{2}) - \sin^2\tfrac{\theta}{2}\ln\sin^2\tfrac{\theta}{2}}$

= sammenfiltringsentropien mellom B og C

Oppgave 3

a) $\vec{\sigma} = \sigma_x \vec{e}_x + \sigma_y \vec{e}_y + \sigma_z \vec{e}_z$

$$= \begin{pmatrix} \vec{e}_z & \vec{e}_x - i\vec{e}_y \\ \vec{e}_x + i\vec{e}_y & -\vec{e}_z \end{pmatrix}$$

$$\vec{\sigma}_{BA} = (0\ 1)\left(- - -\right)\begin{pmatrix}1\\0\end{pmatrix} = \vec{e}_x + i\vec{e}_y \equiv \vec{e}_+$$

$$(\vec{k} \times \vec{\varepsilon}_{ka}) \cdot \vec{e}_+ = (\vec{e}_+ \times \vec{k}) \cdot \vec{\varepsilon}_{ka}$$

$$\vec{k} = k(\cos\varphi \sin\theta\, \vec{e}_x + \sin\varphi \sin\theta\, \vec{e}_y + \cos\theta\, \vec{e}_z)$$

$$\Rightarrow \vec{e}_+ \times \vec{k} = ik(\cos\theta\, \vec{e}_+ - e^{i\varphi}\sin\theta\, \vec{e}_z)$$

Vinkelavhengighet til $|\langle B\, 1_{ka} | \hat{H}_1 | A, 0\rangle|^2$:

$$p(\theta,\varphi) = N \sum_a |(\vec{e}_+ \times \vec{k}) \cdot \vec{\varepsilon}_{ka}|^2 \qquad N \text{ norm. faktor}$$

$$= N\left(|\vec{e}_+ \times \vec{k}|^2 - \underbrace{|(\vec{e}_+ \times \vec{k}) \cdot \tfrac{\vec{k}}{k}|^2}_{=0}\right)$$

$$= Nk^2(2\cos^2\theta + \sin^2\theta) \qquad |\vec{e}_+|^2 = 2$$

$$= Nk^2(1 + \cos^2\theta) \quad \text{uavh av } \varphi$$

Normering $\int d\varphi \int d\theta \sin\theta (1 + \cos^2\theta) = 2\pi \int_{-1}^{1} (1 + u^2)\, du = 2\pi \left[u + \tfrac{1}{3}u^3\right]_{-1}^{1}$

$$= \frac{16}{3}\pi$$

$$\Rightarrow \underline{p(\theta,\varphi) = \frac{3}{16\pi}(1 + \cos^2\theta)}$$

b) $\vec{k} = k\vec{e}_x$

Sannsynlighet for deteksjon av foton med polarisasjon i retning $\vec{\varepsilon}(\alpha)$, $\qquad \vec{e}_+ \times \vec{e}_x = -i\vec{e}_z$

$$p(\alpha) = N' |(\vec{e}_+ \times \vec{e}_x) \cdot \vec{\varepsilon}(\alpha)|^2$$

$$= N' |\vec{e}_z \cdot \vec{\varepsilon}(\vec{\alpha})|^2$$

$$= N' \sin^2\alpha$$

$$p(\alpha) + p(\alpha + \tfrac{\pi}{2}) = N' = 1 \quad \Rightarrow \quad \underline{p(\alpha) = \sin^2\alpha}$$

Sannsynlighet for deteksjon:

$p(0) = 0 \qquad \alpha = 0 \Rightarrow \vec{\varepsilon} = \vec{e}_y$

$p(\frac{\pi}{2}) = 1 \qquad \alpha = \frac{\pi}{2} \Rightarrow \vec{\varepsilon} = \vec{e}_z$

viser at fotoner utsendt langs x-aksen er polarisert langs z-aksen

FYS4110, Exam 2014

Solutions

Problem 1

a) $\hat{\rho}_I = \cos^2 x\, |1\rangle\langle 1| + \sin^2 x\, |2\rangle\langle 2| + \cos x \sin x\, (|1\rangle\langle 2| + |2\rangle\langle 1|)$

$$= \tfrac{1}{2}\cos^2 x\, (|+-\rangle\langle +-| + |-+\rangle\langle -+| + |+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$+ \tfrac{1}{2}\sin^2 x\, (|+-\rangle\langle +-| + |-+\rangle\langle -+| - |+-\rangle\langle -+| - |-+\rangle\langle +-|)$$

$$+ \tfrac{1}{2}\cos x \sin x\, (|+-\rangle\langle +-| - |-+\rangle\langle -+| - |+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$+ \tfrac{1}{2}\cos x \sin x\, (|+-\rangle\langle +-| - |-+\rangle\langle -+| + |+-\rangle\langle -+| - |-+\rangle\langle +-|)$$

$$= \underline{\tfrac{1}{2}(1+\sin(2x))|+-\rangle\langle +-| + \tfrac{1}{2}(1-\sin(2x))|-+\rangle\langle -+|}$$

$$\underline{+ \tfrac{1}{2}\cos 2x\, (|+-\rangle\langle -+| + |-+\rangle\langle +-|)}$$

Reduced density operators

$$\hat{\rho}_{IA} = \mathrm{Tr}_B\, \hat{\rho}_I = \tfrac{1}{2}(1+\sin(2x))\,|+\rangle\langle +| + \tfrac{1}{2}(1-\sin(2x))\,|-\rangle\langle -|$$

$$= \underline{\tfrac{1}{2}(\mathbb{1} + \sin(2x)\,\sigma_z)}$$

$$\hat{\rho}_{IB} = \mathrm{Tr}_A\, \hat{\rho}_I = \tfrac{1}{2}(1-\sin(2x))\,|+\rangle\langle +| + \tfrac{1}{2}(1+\sin(2x))\,|-\rangle\langle -|$$

$$= \underline{\tfrac{1}{2}(\mathbb{1} - \sin(2x)\,\sigma_z)}$$

Entropies: $S_I = 0$ (pure state)

$$S_{IA} = S_{IB} = \underline{-\tfrac{1}{2}(1+\sin(2x))\log\left(\tfrac{1}{2}(1+\sin(2x))\right) - \tfrac{1}{2}(1-\sin(2x))\log\left(\tfrac{1}{2}(1-\sin(2x))\right)}$$

$x = 0, \frac{\pi}{2}$ $\quad S_{IA} = S_{IB} = \log 2$; maximally entangled states

$x = \frac{\pi}{4}$ $\quad S_{IA} = S_{IB} = 0$, non-entangled, product state $|\psi\rangle = |+\rangle \otimes |-\rangle$

b) Case II

$$\hat{\rho}_{II} = \cos^2 x\,|1\rangle\langle 1| + \sin^2 x\,|2\rangle\langle 2|$$

$$\Rightarrow S_{II} = \underline{-\cos^2 x \log(\cos^2 x) - \sin^2 x \log(\sin^2 x)}$$

$\hat{\rho}_{II}$ obtained from $\hat{\rho}_I$ by deleting terms proportional to $\cos x \sin x = \frac{1}{2}\sin(2x)$:

$$\hat{\rho}_{II} = \frac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|) + \frac{1}{2}\cos(2x)(|+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$\Rightarrow \underline{\hat{\rho}_{IIA} = \hat{\rho}_{IIB} = \frac{1}{2}\mathbb{1}} \Rightarrow \underline{S_{IIA} = S_{IIB} = \log 2}$$

$x = 0, \pi/2$ Same as in case I

$x = \pi/4$, $S_{II} = \log 2$; maximally mixed

$\hat{\rho}_{II} = \frac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|)$

separable (sum of product states) $\Rightarrow$ non-entangled

c) $\Delta_I = -S_{IA} = -S_{IB}$

is negative, unless $S_{IA} = S_{IB} = 0$,

which happens for $x = \pi/4$.

$\Delta_{II} = S_{II} - \log 2$

$S_{II} \leq \log 2$ since the Hilbert space is two-dimensional

$\Rightarrow \Delta_{II} \leq 0$, $\Delta_{II} = 0$ only when $S_{II} = \log 2$,

this happens only when $\underline{x = \pi/4}$ $\Rightarrow \cos^2 x = \sin^2 x = \frac{1}{2}$

Problem 2

a) Matrix elements of $\hat{x}$

$$X_{mn} = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle m|\hat{a}^{+}|n\rangle + \langle m|\hat{a}|n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right)$$

Non-vanishing: $X_{n-1,n} = X_{n,n-1} = \sqrt{\frac{\hbar n}{2m\omega}}$

Photon emission: $|n\rangle \rightarrow |n-1\rangle \quad (E_n \rightarrow E_{n-1} + \hbar\omega)$

$$\Rightarrow \underline{W_{n-1,n} = \frac{2\alpha\hbar}{3mc^2}\omega^2 n = \gamma n}$$

b) $$\frac{dp_n}{dt} = \langle n|\left(-\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] - \frac{1}{2}\gamma(\hat{a}^{+}\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^{+}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^{+})\right)|n\rangle$$

$$= \underline{-\gamma\left(np_n - (n+1)p_{n+1}\right)}$$

$W_{n-1,n}$ = transition rate when state $|n\rangle$ occupied

$\Rightarrow p_n = 1,\ p_m = 0 \quad m \neq n$

With this assumption, conservation of probability gives $\frac{dp_n}{dt} = -W_{n-1,n}$

$= -\gamma n$ (from eq. (9))

consistent with eq. (8).

c) Excitation energy

$$E = Tr(\hat{H}_o \hat{\rho}) - \frac{1}{2}\hbar\omega$$

$$= \sum_n \hbar\omega(n+\tfrac{1}{2})\langle n|\hat{\rho}|n\rangle - \tfrac{1}{2}\hbar\omega$$

$$= \sum_n \hbar\omega\, n\, p_n$$

$$\Rightarrow \frac{dE}{dt} = \hbar\omega \sum_n n \frac{dp_n}{dt}$$

$$= -\gamma\hbar\omega \sum_n (n^2 p_n - n(n+1)p_{n+1})$$

$$= -\gamma\hbar\omega \sum_n (n^2 - n(n-1))\, p_n$$

$$= -\gamma\hbar\omega \sum_n n\, p_n$$

$$= \underline{-\gamma E}$$

Integrated

$$\frac{dE}{E} = -\gamma\, dt \Rightarrow \ln E = -\gamma t + \text{const}$$

$$\Rightarrow \underline{E(t) = E(0)\, e^{-\gamma t}}$$ exponential decay

Problem 3

a) $\mathrm{Tr}\,\hat{\rho} = 1 \Rightarrow N(\beta)^{-1} = \mathrm{Tr}(e^{-\beta H})$

$$= \sum_n e^{-\beta E_n}$$

$$E(\beta) = \mathrm{Tr}(\hat{H}\hat{\rho}) = N\,\mathrm{Tr}(\hat{H} e^{-\beta \hat{H}})$$

$$= -N \frac{\partial}{\partial \beta} \mathrm{Tr}(e^{-\beta \hat{H}}) = -N \frac{\partial}{\partial \beta} N^{-1}$$

$$= \frac{1}{N} \frac{\partial}{\partial \beta} \ln N = \underline{\frac{\partial}{\partial \beta} \ln N(\beta)}$$

Entropy: $S(\beta) = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho})$

$$= -\mathrm{Tr}(N e^{-\beta \hat{H}} (\ln N - \beta \hat{H}))$$

$$= -\ln N\,\mathrm{Tr}\,\hat{\rho} + \beta\,\mathrm{Tr}(\hat{H}\hat{\rho})$$

$$= -\ln N + \beta E(\beta)$$

$$= \underline{\beta \frac{\partial}{\partial \beta} \ln N(\beta) - \ln N(\beta)}$$

b) $\hat{H} = \frac{1}{2} \varepsilon \sigma_z \Rightarrow E_\pm = \pm \frac{1}{2} \varepsilon$

$$\Rightarrow N^{-1} = e^{\frac{1}{2}\varepsilon\beta} + e^{-\frac{1}{2}\varepsilon\beta} = 2\cosh(\tfrac{1}{2}\varepsilon\beta)$$

$$N(\beta) = \underline{\frac{1}{2\cosh(\frac{1}{2}\varepsilon\beta)}}$$

$$E(\beta) = -2\cosh(\tfrac{1}{2}\varepsilon\beta) \frac{1}{2\cosh^2(\frac{1}{2}\varepsilon\beta)} \sinh(\tfrac{1}{2}\varepsilon\beta) \cdot \tfrac{1}{2}\varepsilon$$

$$= \underline{-\tfrac{1}{2}\varepsilon \tanh(\tfrac{1}{2}\varepsilon\beta)}$$

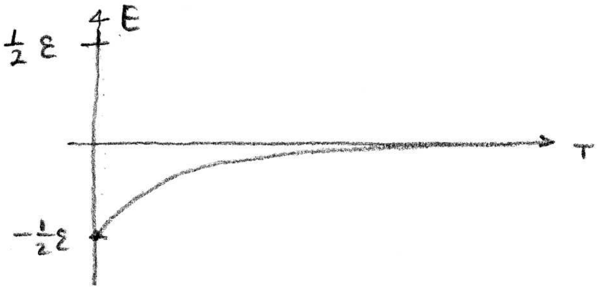
$$\underline{S(\beta) = -\tfrac{1}{2}\varepsilon\beta \tanh(\tfrac{1}{2}\varepsilon\beta) + \ln(2\cosh(\tfrac{1}{2}\varepsilon\beta))}$$

$$E(\beta) = -\tfrac{1}{2}\varepsilon \tanh(\tfrac{1}{2}\varepsilon\beta)$$

$$= -\tfrac{1}{2}\varepsilon \frac{e^{\frac{1}{2}\varepsilon\beta} - e^{-\frac{1}{2}\varepsilon\beta}}{e^{\frac{1}{2}\varepsilon\beta} + e^{-\frac{1}{2}\varepsilon\beta}}$$

$$T \to 0 \Rightarrow \beta \to \infty \Rightarrow E(\beta) \simeq -\tfrac{1}{2}\varepsilon(1 - e^{-\varepsilon\beta}) \to -\tfrac{1}{2}\varepsilon$$

$$T \to \infty \Rightarrow \beta \to 0 \Rightarrow E(\beta) \simeq -\tfrac{1}{4}\varepsilon^2\beta = -\tfrac{1}{4}\frac{\varepsilon^2}{k_\beta T} \to 0$$



c) $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) \Rightarrow \vec{r} = Tr(\vec{\sigma}\hat{\rho})$

since $Tr\,\sigma_i = 0$ and $Tr(\sigma_i\sigma_j) = 2\delta_{ij}$

$$\vec{r} = N\, Tr(\vec{\sigma}\, e^{-\frac{1}{2}\varepsilon\beta\sigma_z})$$

$$= N\, Tr(\sigma_z e^{-\frac{1}{2}\varepsilon\beta\sigma_z})\,\vec{k}$$

$$= -\tfrac{2}{\varepsilon} N \frac{\partial}{\partial\beta}(Tr\, e^{-\frac{1}{2}\varepsilon\beta\sigma_z})\,\vec{k}$$

$$= -\tfrac{2}{\varepsilon} N \frac{\partial}{\partial\beta} N^{-1}\vec{k}$$

$$= -\tfrac{2}{\varepsilon} E(\beta)\,\vec{k}$$

$$= \underline{\tanh(\tfrac{1}{2}\varepsilon\beta)\,\vec{k}}$$

$\vec{r} = r\vec{k}$ with $r = -\frac{2}{\varepsilon}E(\beta)$

$T = 0$ $(\beta = \infty)$: $r = 1$ pure state

$T \to \infty$ $(\beta \to 0)$: $r \to 0$ maximally mixed