9.1 Two-level systems

$$\hat{H} = \frac{1}{2}\hbar\omega(\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) - i\hbar\lambda(\sigma_+ \otimes \sigma_- - \sigma_- \otimes \sigma_+)$$

a) $|\psi(t)\rangle = \cos\lambda t\, |+-\rangle + \sin\lambda t\, |-+\rangle$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = i\hbar(-\lambda \sin\lambda t\, |+-\rangle + \lambda\cos\lambda t\, |-+\rangle)$$

$$\hat{H}|\psi(t)\rangle = \frac{1}{2}\hbar\omega\big((+1-1)\cos\lambda t\, |+-\rangle + (-1+1)\sin\lambda t\, |-+\rangle\big)$$
$$-i\hbar\lambda(-\cos\lambda t\, |-+\rangle + \sin\lambda t\, |+-\rangle) = i\hbar\frac{d}{dt}|\psi(t)\rangle$$

$\Rightarrow |\psi(t)\rangle$ is a solution.

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \cos^2\lambda t\, |+-\rangle\langle+-| + \sin^2\lambda t\, |-+\rangle\langle-+|$$
$$+ \sin\lambda t\cos\lambda t\,(|+-\rangle\langle-+| + |-+\rangle\langle+-|)$$

b) $|\pm\rangle\langle\pm| = \frac{1}{2}(\mathbb{1} \pm \sigma_z)$, $|\pm\rangle\langle\mp| = \sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$

$$\hat{\rho} = \frac{1}{4}\big(\cos^2\lambda t(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z$$
$$+ \sin^2\lambda t(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z$$
$$+ 2\sin\lambda t\cos\lambda t(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y)\big)$$
$$= \frac{1}{4}\big(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\sigma_z + \cos 2\lambda t(\sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z) + \sin 2\lambda t(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y)\big)$$

$\mathrm{Tr}\,\sigma_i = 0$, $\mathrm{Tr}\,\mathbb{1} = 2$

$$\hat{\rho}_A = \mathrm{Tr}_B\,\hat{\rho} = \frac{1}{4}(\mathbb{1}\cdot 2 + \cos 2\lambda t\,\sigma_z \cdot 2) = \frac{1}{2}(\mathbb{1} + \cos 2\lambda t\,\sigma_z)$$
$$\hat{\rho}_B = \mathrm{Tr}_A\,\hat{\rho} = \frac{1}{4}(2\mathbb{1} - 2\cos 2\lambda t\,\sigma_z) = \frac{1}{2}(\mathbb{1} - \cos 2\lambda t\,\sigma_z)$$

c) Eigenvalues of $\hat{\rho}_A, \hat{\rho}_B$ are $\frac{1 \pm \cos 2\lambda t}{2}$, or $\sin^2\lambda t$ and $\cos^2\lambda t$

$$S = -\mathrm{Tr}\,\hat{\rho}_A \log\hat{\rho}_A = -\cos^2\lambda t \log\cos^2\lambda t - \sin^2\lambda t \log\sin^2\lambda t$$

9.2 Distributed information

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}\left(|+--\rangle + \eta^n|-+-\rangle + \eta^{*n}|--+\rangle\right)$$

$$\eta = e^{\frac{2\pi i}{3}} \qquad (\eta^* = \eta^{-1} = \eta^2)$$

a) $$\hat{\rho} = |\psi_n\rangle\langle\psi_n| = \frac{1}{3}\Big(|+--\rangle\langle+--| + |-+-\rangle\langle-+-| + |--+\rangle\langle--+|$$
$$+\eta^{-n}|+--\rangle\langle-+-| + \eta^{n}|-+-\rangle\langle+--| + \eta^{n}|+--\rangle\langle--+|$$
$$+\eta^{-n}|--+\rangle\langle+--| + \eta^{n}|-+-\rangle\langle--+| + \eta^{n}|--+\rangle\langle-+-|\Big)$$

$\hat{\rho}_A = \mathrm{Tr}_{BC}\hat{\rho} = \frac{1}{3}\left(|+\rangle\langle+| + 2|-\rangle\langle-|\right)$ is independent of n.

$\mathrm{Tr}\left(\hat{\rho}\,|\pm_A \pm_B \pm_C\rangle\langle\pm_A \pm_B \pm_C|\right) = \frac{1}{3}$ for $|+--\rangle$, $|-+-\rangle$, $|--+\rangle$, otherwise, 0.
The outcome probabilities are n-independent and no information about n can be extracted this way.

b) $$\hat{\rho}_{AB} = \frac{1}{3}\left(|+-\rangle\langle+-| + |-+\rangle\langle-+| + |--\rangle\langle--| + \eta^{-n}|+-\rangle\langle-+| + \eta^{n}|-+\rangle\langle+-|\right)$$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
Implicit tensor products:

$$\hat{\rho}_{AB} = \frac{1}{12}\Big((|0\rangle+|1\rangle)(|0\rangle-|1\rangle)(\langle0|+\langle1|)(\langle0|-\langle1|)$$
$$+(|0\rangle-|1\rangle)(|0\rangle+|1\rangle)(\langle0|-\langle1|)(\langle0|+\langle1|)$$
$$+(|0\rangle-|1\rangle)(|0\rangle-|1\rangle)(\langle0|-\langle1|)(\langle0|-\langle1|)$$
$$+\eta^{-n}(|0\rangle+|1\rangle)(|0\rangle-|1\rangle)(\langle0|-\langle1|)(\langle0|+\langle1|)$$
$$+\eta^{n}(|0\rangle-|1\rangle)(|0\rangle+|1\rangle)(\langle0|+\langle1|)(\langle0|-\langle1|)\Big)$$
$$= \frac{1}{12}\Big((3+\eta^n+\eta^{-n})(|00\rangle\langle00| + |11\rangle\langle11|)$$
$$+(3-\eta^n-\eta^{-n})(|01\rangle\langle01| + |10\rangle\langle10|)$$
$$+\ldots\ (|10\rangle\langle01|,\ |00\rangle\langle01|\ \text{etc.})\Big)$$

9.2

... b) $\eta^{1} + \eta^{-1} = \eta^{2} + \eta^{-2} = -1 \Rightarrow P(k|1) = P(k|2)$

$p(k|n) = \langle k | \rho_{AB}^{(n)} | k \rangle$

$P(1|0) = \frac{3+1+1}{12} = \underline{\frac{5}{12}}$ $\quad P(1|1) = \frac{3-1}{12} = \underline{\frac{1}{6}}$

$P(2|0) = \frac{3-1-1}{12} = \underline{\frac{1}{12}}$ $\quad P(2|1) = \frac{3+1}{12} = \underline{\frac{1}{3}}$

c) $P(1|0) = \frac{5}{12}$, $P(1|1) = \frac{1}{6} = P(1|2)$

$\bar{p}(n|k) = \frac{p(k|n)}{p(k)}$, $\sum_n \bar{p}(n|k) = 1 \Rightarrow p(k) = \sum_n p(k|n) = \frac{9}{12}$

$p(k)$ is the normalization constant. In terms of probabilities, it is $P(k)/P(n)$.

$\bar{p}(0|1) = \underline{\underline{\frac{5}{9}}}$, $\bar{p}(1|1) = \bar{p}(2|1) = \underline{\underline{\frac{2}{9}}}$

$n=0$ is most likely.