UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS4110/9110 Modern Quantum Mechanics
Day of exam: Monday 5 December 2016
Exam hours: 4 hours, beginning at 14:30
This examination paper consists of 3 problems, written on 3 pages
Permitted materials: Approved calculator
Angell and Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Language: The solutions may be written in Norwegian or English depending on your own preference. *Note: This paper is also available in norsk bokmål and nynorsk.*

Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1
Spin-half particle in a harmonic oscillator potential
A spin-half particle is moving in a one-dimensional harmonic oscillator potential (in the x-direction) under the influence of a constant magnetic field (in the z-direction). The Hamiltonian is

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_1\sigma_z + \lambda\hbar(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{1}$$

where the first term is the harmonic oscillator part, with ω_0 as the oscillator angular frequency, and the second term is the spin energy due to the magnetic field, with ω_1 as the angular spin precession frequency. The third term is a coupling term between the spin and the position coordinates of the particle, with λ as a coupling parameter. The spin flip operators are defined as $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, $\hat{a}$, $\hat{a}^\dagger$ are the standard lowering and raising operators of the harmonic oscillator, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin matrices.

When $\lambda = 0$, the spin and position of the particle are uncoupled and the energy eigenstates are $|n, m\rangle$, with $n = 0, 1, 2, ...$ as the harmonic oscillator quantum number and $m = \pm 1$ as the spin quantum number, corresponding to spin up/down along the z-axis. When $\lambda \neq 0$, the unperturbed eigenstates will pairwise be coupled by the Hamiltonian, so that $|n, +1\rangle$ is coupled to $|n+1, -1\rangle$. (The state $|0, -\rangle$ is an exception; it is not affected by the coupling term and remains the non-degenerate ground state also for $\lambda \neq 0$.)

a) Consider the two-dimensional subspace spanned by the basis vectors $|0, +1\rangle$ and $|1, -1\rangle$. Show that in this space, and in the given basis, the Hamiltonian takes the form of a 2x2 matrix

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \hbar\epsilon\mathbb{1} \tag{2}$$

with $\mathbb{1}$ as the 2x2 identity matrix. Determine Δ, $\cos\theta$, $\sin\theta$ and ϵ.

b) Find the energies and eigenstates of H in the two-dimensional subspace, expressed as functions of Δ, θ and ϵ.

c) The basis vectors $|n,m\rangle$ can be regarded as tensor products of position and spin vectors, $|n,m\rangle = |n\rangle \otimes |m\rangle$. The two eigenstates found under b) will be entangled with respect to the position and spin variables. Determine the entanglement entropy as function of θ. What value for θ gives the least and what gives the greatest entanglement?

PROBLEM 2
Coupled harmonic oscillators

Two harmonic oscillators, referred to as $\mathcal{A}$ and $\mathcal{B}$, form a composite quantum mechanical system. The Hamiltonian of the system has the form

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \mathbb{1}) + \hbar\lambda(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \tag{3}$$

with $(\hat{a}, \hat{a}^\dagger)$ as lowering and raising operators for $\mathcal{A}$ and $(\hat{b}, \hat{b}^\dagger)$ as corresponding operators for $\mathcal{B}$, while ω and λ are real valued constants.

a) Show that the Hamiltonoperator can be expressed in diagonal form as

$$\hat{H} = \hbar\omega_c\,\hat{c}^\dagger\hat{c} + \hbar\omega_d\,\hat{d}^\dagger\hat{d} + \hbar\omega\mathbb{1} \tag{4}$$

where c and d are linear combinations of a and b,

$$\hat{c} = \mu\,\hat{a} + \nu\,\hat{b}\,, \quad \hat{d} = -\nu\,\hat{a} + \mu\,\hat{b} \tag{5}$$

with μ and ν as real constants satisfying $\mu^2 + \nu^2 = 1$, and determine the new parameters μ, ν, ω_c, and ω_d, expressed in terms of ω og λ. (The same type of expressions are found for the harmitian conjugate operators $\hat{c}^\dagger$ og $\hat{d}^\dagger$.) Check that the new operators $\hat{c}$ and $\hat{d}$ satisfy the same set of harmonic oscillator commutation relations as $\hat{a}$ and $\hat{b}$. It is sufficient to show

$$\left[\hat{c}, \hat{c}^\dagger\right] = \left[\hat{d}, \hat{d}^\dagger\right] = \mathbb{1}\,, \quad \left[\hat{c}, \hat{d}^\dagger\right] = 0 \tag{6}$$

b) Assume that the state $|\psi(0)\rangle$ of the composite system, at time $t = 0$, is a coherent state when expressed in terms of the new variables,

$$\hat{c}\,|\psi(0)\rangle = z_{c\,0}|\psi(0)\rangle\,, \quad \hat{d}\,|\psi(0)\rangle = z_{d\,0}|\psi(0)\rangle \tag{7}$$

Also at a later time the state $|\psi(t)\rangle$ will be a coherent state for both $\hat{c}$ og $\hat{d}$, with eigenvalues

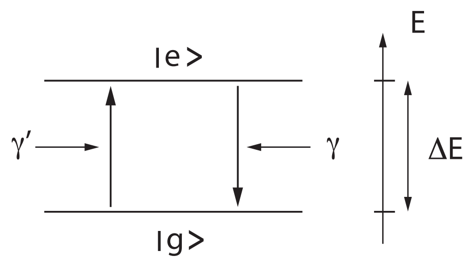
$$z_c(t) = e^{-i\omega_c t}z_{c\,0}\,, \quad z_d(t) = e^{-i\omega_d t}z_{d\,0} \tag{8}$$

Show this for $z_c(t)$. (The expression for $z_d(t)$ follows in the same way, and is therefore not needed to be shown.)

c) Show that the state $|\psi(t)\rangle$ is a coherent state also for the original harmonic oscillator operators $\hat{a}$ og $\hat{b}$, and find the eigenvalues $z_a(t)$ and $z_b(t)$ expressed in terms of $z_{a\,0}$and $z_{b\,0}$.

PROBLEM 3
Two-level system in a heat bath

We consider a two-level system, with $|g\rangle$ as the ground state and $|e\rangle$ as the excited state of the Hamiltonian $\hat{H}_0$. The energy difference between the corresponding two energy levels is ΔE.



The system interacts weakly with a heat bath with temperature T. Energy can flow both ways, with γ as the rate for emission of energy to the heat bath in the transition $|e\rangle \to |g\rangle$ and γ' as the rate for absorption of energy in the transition $|g\rangle \to |e\rangle$. The situation is illustrated in the figure.

The temperature T of the heat bath and the energy gap ΔE determine the ratio between γ' and γ,

$$\gamma' = \gamma\, e^{-\Delta E/kT} \tag{9}$$

where k is the Boltzman constant. Both transitions, corresponding to γ and γ', contribute to the time evolution of the density operator of the two-level system. This is expressed by the Lindblad equation in the following way,

$$\begin{aligned}\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] \quad &- \quad \frac{1}{2}\gamma(|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e| - 2|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|) \\ &- \quad \frac{1}{2}\gamma'(|g\rangle\langle g|\hat{\rho} + \hat{\rho}|g\rangle\langle g| - 2|e\rangle\langle g|\hat{\rho}|g\rangle\langle e|)\end{aligned} \tag{10}$$

The 2×2 matrix form of $\hat{\rho}$, in the basis $\{|g\rangle, |e\rangle\}$, we write as

$$\rho = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{11}$$

with p_e interpreted as the probability of occupation of the excited level and p_g as the probability of occupation of the ground state.

a) Find from equation (10) expressions for the time derivatives of p_e, p_g and b, and check that they are consistent with preservation of total probability, $p_e + p_g$.

b) The conditions for $\hat{\rho}$ to be a density operator give restrictions on the matrix elements in (11). What are these?

c) Assume first that the two-level system and the heat bath are in thermal equilibrium, and the density matrix (11) therefore is time independent. Determine the values of variables p_e, p_g and b in this case.

d) Consider next the situation with initial values $p_g = 1$, $p_e = 0$. Determine the time evolution of the occupation probabilities towards thermal equilibrium. What happens in the limits $T \to 0$ and $T \to \infty$?