FYS 4110/9110 Modern Quantum Mechanics

Midterm Exam, Fall Semester 2016

Return of solutions
The problem set is available from Monday morning, 17 October.
Written/printed solutions should be returned to Ekspedisjonskontoret in the Physics Building before Monday, 24 October, at 12:00.
Use candidate numbers rather than full names.

Language
Note: The problem set is available also in Norwegian.
Solutions may be written in Norwegian or English depending on your preference.

Questions concerning the problems
Please ask Jon Magne Leinaas (room Ø471), or Paul Bätzing (room V316, or on the Piazza page).

The problem set consists of 2 problems written on 5 pages.

PROBLEMS

1 Entangled photons
In this problem correlations between pairs of entangled photons are studied. The interesting degree of freedom is the photon polarization. For a single photon the polarization corresponds to a quantum state vector in a two-dimensional Hilbert space spanned by the vectors $|H\rangle$ and $|V\rangle$. These vectors correspond to linear polarization in the horizontal and vertical direction, respectively. A general polarization state is a linear combination of these two. As special cases we consider linearly polarized photons in rotated directions,

$$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{1}$$

The two-photon states, when only polarization is taken into account, are vectors in the tensor product space spanned by the four vectors,

$$\begin{aligned} |HH\rangle &= |H\rangle \otimes |H\rangle \,, \quad |HV\rangle = |H\rangle \otimes |V\rangle \,, \\ |VH\rangle &= |V\rangle \otimes |H\rangle \,, \quad |VV\rangle = |V\rangle \otimes |V\rangle \,, \end{aligned} \tag{2}$$

(Note that even if the photons are bosons there is no symmetry constraint on the two-photon states, since we assume that the two photons can be distinguished by their different direction of propagation.)

As a specific way to produce entangled photon pairs we consider the method of *parametric down conversion*, as outline below and sketched in Figs. 2 and 3. As illustrated in Fig. 2a a beam of photons enters a crystal, where single photons, due to the non-linear interaction with the crystal, are split into pairs of photons, which carry half the energy of the incoming photon. The transverse momentum of the emerging photons is fixed so that their direction of propagation is limited to a cone, as indicated in the figure. The photons appear with constant probability around the cone. However, due to conservation of total transverse momentum, the two photons in each a pair are correlated so that they always are emitted at opposite sides of the cone.

There is furthermore a polarization effect, since photons with horizontal and vertical polarization (relative to the crystal planes) do not propagate in exactly the same way. As a consequence the

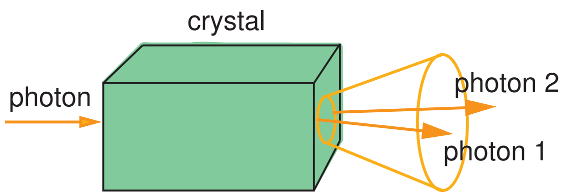

Fig. 2a

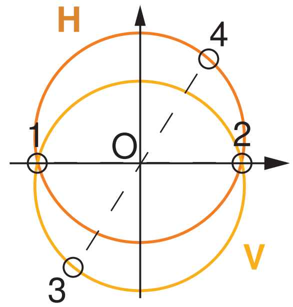

Fig. 2b

cones corresponding to these two polarizations are slightly shifted. This is shown in the head-on view of Fig. 2b, where the cone corresponding to polarization H is slightly lifted relative to the cone corresponding to polarization V.

Two photons in a correlated pair will be located on opposite points of the central point O, like the pair of points 1 and 2 and the pair 3 and 4, and they always appear with orthogonal polarization. As shown by the figure this means that for most directions of the emitted photons the polarization of each photon is uniquely determined by its direction of propagation. For such a pair the two-photon state is a product state of the form $|HV\rangle = |H\rangle \otimes |V\rangle$. As an illustration, the pair $3, 4$ of directions of the cone, as shown in Fig.2b, will be of this type.

However two directions are unique since they lie on both cones. This is illustrated by the points 1 and 2 in Fig. 2b. A photon at one of these positions will be in a superposition of $|H\rangle$ and $|V\rangle$. Due to correlations between the photons a pair located at these positions will be described by an entangled two-photon state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + e^{i\chi}|VH\rangle) \tag{3}$$

where the complex phase χ can be regulated in the experimental set up.

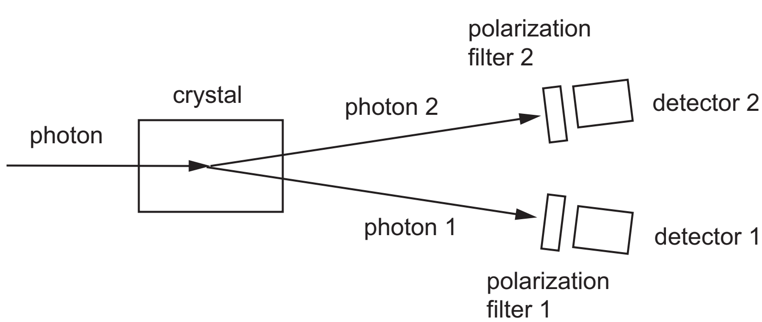

Fig. 3

The experimental set up is schematically shown in Fig. 3. It is assumed that only pairs of entangled photons are filtered out in the beams that reach the two detectors. To analyze correlations between the two photons, polarization filters are applied to photons in both directions, as shown in the figure.

Those that pass the polarization filters are registered in the detectors and the registrations are paired by use of coincidence counters. We assume idealized conditions, by disregarding experimental errors.

The polarization filters may be represented by operators that project on linearly polarized states along rotated directions

$$\hat{P}(\theta) = |\theta\rangle\langle\theta|\,, \quad |\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle \tag{4}$$

In the following we examine the expected results of the polarization measurements by calculating the following expectation values

$$\begin{aligned} P_1(\theta_1) &\equiv \left\langle \hat{P}_1(\theta_1) \right\rangle && \text{photon 1} \\ P_2(\theta_2) &\equiv \left\langle \hat{P}_2(\theta_2) \right\rangle && \text{photon 2} \\ P_{12}(\theta_1,\theta_2) &\equiv \left\langle \hat{P}_1(\theta_1) \otimes \hat{P}_2(\theta_2) \right\rangle && \text{photon 1 and photon 2} \end{aligned} \tag{5}$$

a) Assume that the photon beam produces N entangled photon pairs in a given time interval. In this time interval n_1 photons are registered in detector 1, n_2 photons are registered in detector 2 and and n_{12} are registered at coincidence in the two detectors. What are the relations between the frequences n_1/N, etc. and the the expectation values P_1, P_2 and P_{12}?

b) For the general two-photon state of the form (3) find the density operator of the two-photon pair, and the corresponding reduced density operators for photon 1 and photon 2. Characterize the degree of entanglement of the two photons.

We consider now three different situations where the the entangled photon pairs are produced in the states (3) with I: $\chi = \pi$, II: $\chi = 0$ and III: $\chi = \pi/2$.

c) For all the three cases I, II and III, determine $P_1(\theta_1)$, $P_2(\theta_2)$, and $P_{12}(\theta_1, \theta_2)$.

d) Show that there exists a separable state, in the form of a probabilistic mixture of $|HV\rangle$ and $|VH\rangle$, which has identical expectation values to those in case III.

e) The Bell inequality, which is based on an assumed set of "hidden variables" as a source of the statistical distributions, can be written as a constraint on the function P_{12} in the following way (see Sect. 2.3.2 of the lecture notes),

$$F(\theta_1,\theta_2,\theta_3) \equiv P_{12}(\theta_2,\theta_3) - |P_{12}(\theta_1,\theta_2) - P_{12}(\theta_1,\theta_3)| \geq 0 \tag{6}$$

Examine the Bell inequality in the cases I, II and III for the special choice of angles $\theta_1 = 0$, $\theta_2 = \theta$ and $\theta_3 = 2\theta$ by plotting $F(0, \theta, 2\theta)$ as a function of θ. Based on the plots comment on whether the Bell inequality is satisfied or not and show in particular that in case III Bell's inequality is *not* broken. Is there a relation between this conclusion for case III and the results in d)?

For entangled photons one would expect that one will be able to detect breaking of Bell's inequality. However, in case III, this seems not to bee the case. A possible explanation may be that this is due to the restriction of the two analyzers to *linear* polarization. To investigate this one of the analyzer is changed with the new polarization states and projection operators

$$\hat{P}(\theta_\phi) = |\theta_\phi\rangle\langle\theta_\phi|\,, \quad |\theta_\phi\rangle = \frac{1}{\sqrt{2}}(e^{i\phi}\cos\theta|H\rangle + e^{-i\phi}\sin\theta|V\rangle) \tag{7}$$

where we restrict ϕ to the two cases $\phi = \pi/4$ and $\phi = -\pi/4$, which correspond to circular polarization, either left-handed or right-handed.

f) Consider a similar experimental set up as before, with detector 1 having an unchanged filter, which projects on states of the form (4), while detector 2 now is projecting on the new states (7). We distinguish between the two cases A: $\phi = \pi/4$ and B: $\phi = -\pi/4$. Determine also in these two cases the joint probability $P_{12}(\theta_1, \theta_2)$ and show that in these cases Bell's inequality is broken. Make a comparison with the earlier cases I-III.

2 Atom-photon interactions in a microcavity

An atom is trapped inside a small reflecting cavity. The energy difference between the ground state and the first excited state is $\Delta E = \hbar\omega$, with ω matching the frequency of one of the modes of the electromagnetic field in the cavity. This gives a strong coupling between the atomic states and this field mode, while the couplings to the other cavity modes are weak and can be neglected.

The composite system, the atom plus the resonant cavity mode, is described by the following effective Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) - i\gamma\hbar a^\dagger a \tag{8}$$

where the Pauli matrices act between the two atomic levels, with σ_z being diagonal in the energy basis, and $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ being matrices that raise or lower the atomic energy. $\hat{a}^\dagger$ and $\hat{a}$ are the photon creation and destruction operators, λ is an interaction parameter and γ is a decay parameter. The decay is due to the process where the photon escapes through the cavity walls. Both λ and γ are real-valued parameters, and we assume $\omega > \lambda > \gamma$.

We characterize the relevant states of the composite system as $|g, 0\rangle$, $|g, 1\rangle$ and $|e, 0\rangle$, where g refers to the atomic ground state, e to the excited state, and 0 and 1 refers to the absence or presence of a photon in the cavity mode.

a) Show that in the two-dimensional subspace spanned by the vectors $|g, 1\rangle$ and $|e, 0\rangle$ the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix} \tag{9}$$

where $|e, 0\rangle$ corresponds to the upper row of the matrix and $|g, 1\rangle$ to the lower one, and $\mathbb{1}$ is the identity matrix.

We define the time evolution operator in the usual way as

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{\hbar}\hat{H}t} \tag{10}$$

with the expression being valid for $t \geq 0$. The Hamiltonian (9) is non-hermitian due to the decay of the cavity field, and therefore the time evolution operator is non-unitary. However, we shall below see how to compensate for this.

b) Show that the time evolution operator can be written as

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}(\cos(\Omega t)\mathbb{1} - i\sin(\Omega t)\,\frac{\mathbf{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}) \tag{11}$$

where $\mathbf{\Omega}$ is a complex vector, with $\Omega^2 \equiv \mathbf{\Omega}^2$ being real and positive. Determine $\mathbf{\Omega}$ and Ω. (Note that $\mathbf{\Omega}^2$ contains no complex conjugation, and should therefore not be confused with $|\mathbf{\Omega}|^2$.) The Pauli matrix $\boldsymbol{\sigma}$ in (11) refers to the 2×2 matrix formulation (9) of $\hat{H}$.

c) Assume the system initially to be in the state $|\psi(0)\rangle = |e, 0\rangle$. Determine the time evolution of the state vector, $|\psi(t)\rangle$.

There is one important defect with the description of the time evolution discussed so far. Since the time evolution operator is non-unitary, the norm of of the state vector $|\psi(t)\rangle$ is not preserved, but decays with time. Something seems thus to be missing in the description, and we shall now correct for that. Let us for this purpose add a contribution to the density operator $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$, to give the full density operator of the atom-photon system in the cavity as

$$\hat{\rho}_{cav}(t) = \hat{\rho}(t) + f(t)|g, 0\rangle\langle g, 0| \tag{12}$$

with the function $f(t)$ defined so that the norm of $\hat{\rho}_{cav}(t)$ is conserved with value 1.

d) Determine function $f(t)$, and comment on in what sense the addition of the last term in (12) is reasonable, when considering the physical process described by the Hamiltonian (9).

e) Determine and plot, in a common diagram, the time dependent occupation probabilities of the two atomic levels, as well as the probability for one photon to be present in the cavity. Use in the plot $\tau = \lambda t$ as dimensionless time parameter, $\gamma/\lambda = 0.1$ as numerical value for the dimensionless decay parameter, and make the plot for a the interval $0 < \tau < 50$.

The transmission of the photon through the walls implies that the atom-photon system in the cavity, which we now consider as one subsystem, is coupled to the electromagnetic field outside the cavity, which we consider as a second subsystem. We make the assumption that the total system, consisting of the two subsystems, is all the time in a pure, but entangled, quantum state.

f) Show that the density operator $\hat{\rho}_{cav}(t)$ of the atom-photon system has two non-vanishing eigenvalues, given by $f(t)$ and $1 - f(t)$, and use this to determine the entanglement entropy of the two subsystems. Make a plot of the time-dependent entanglement entropy in the same time interval as the first plot.