**FYS 4110 Modern Quantum Mechanics, Fall Semester 2016**

## Problem set 10

**10.1 The canonical commutation relations.**

The basic commutation relations of the electromagnetic field can be written as

$$\left[\hat{A}_{\mathbf{k}a}, \hat{E}^{\dagger}_{\mathbf{k}'a'}\right] = -i\frac{\hbar}{\epsilon_0}\delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{1}$$

where $\mathbf{k}$ is the wave vector of the electromagnetic field and $a$ is a polarization index. The photon creation and annihilation operators are related to the field operators by

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(\hat{a}_{\mathbf{k}a} + \hat{a}^{\dagger}_{-\mathbf{k}\bar{a}}), \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}(\hat{a}_{\mathbf{k}a} - \hat{a}^{\dagger}_{-\mathbf{k}\bar{a}}) \tag{2}$$

where $\bar{a}$ is related to $a$ by a permutation of the two values of the polarization index. Use the above expressions to show that the creation and annihilation operators satisfy the standard harmonic oscillator commutation relations,

$$\left[\hat{a}_{\mathbf{k}a}, \hat{a}^{\dagger}_{\mathbf{k}'a'}\right] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{aa'} \tag{3}$$

**10.2 Photon emission**

A particle with mass $m$ and charge $e$ is trapped in a one-dimensional harmonic oscillator potential, with the motion restricted to the $z$-axis. The frequency of the oscillator is $\omega$. At time $t = 0$ the particle is excited to energy level $n$ and it then performs a transition to level $n-1$ by emitting one photon of energy $\hbar\omega$. We write the energy eigenstates of the composite system of charged particle and photons as $|n, n_{\mathbf{k}a}\rangle$. With initially no photon present the state is $|i\rangle = |n, 0\rangle$, while the final state with one photon present is $|f\rangle = |n-1, 1_{\mathbf{k}a}\rangle$. To first order in perturbation theory the angular probability distribution $p(\theta, \phi)$ of the emitted photon is

$$p(\theta,\phi) = \kappa\sum_a |\langle n-1, 1_{\mathbf{k}a}|\hat{H}_{emis}|n, 0\rangle|^2 \tag{4}$$

with $(\theta, \phi)$ as the polar angle of the photon quantum number $\mathbf{k}$ and $\kappa$ as a proportionality factor. $\hat{H}_{emis}$ is the emission part of the interaction Hamiltonian, which in the dipole approximation is

$$\hat{H}_{emis} = -\frac{e}{m}\sum_{\mathbf{k}a}\sqrt{\frac{\hbar}{2V\epsilon_0\,\omega}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\varepsilon}_{\mathbf{k}a}\,\hat{a}^{\dagger}_{\mathbf{k}a} \tag{5}$$

a) Show that for an arbitrary (real) vector $\mathbf{a}$ we have the identity

$$\sum_a(\mathbf{a}\cdot\boldsymbol{\epsilon}_{\mathbf{k}a})^2 = \mathbf{a}^2 - (\mathbf{a}\cdot\frac{\mathbf{k}}{k})^2 \tag{6}$$

b) Determine the particle matrix element $\langle n-1|\hat{\mathbf{p}}|n\rangle$.

c) Find the probability distribution $p(\theta, \phi)$.

The relation between the momentum operator and the ladder operators of the harmonic oscillator is found in Sect. 1.4.4 of the lecture notes.

**10.3 Dressed photon states (Exam 2011)**

A photon is interacting with an atom within a small reflecting cavity. The photon energy is close to the excitation energy of the atom, which connects the ground state to the first excited state. Due to interaction between the photon and the atom the stationary states of the composite system are admixtures of the photon and atomic states. These are sometimes referred to as a ”dressed” photon states. In this problem we examine some of the properties of the dressed states.

The Hamiltonian of the photon-atom system can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar\lambda(\hat{a}^\dagger\sigma_- - \hat{a}\sigma_+) \tag{7}$$

where $\hbar\omega_0$ is then the energy difference between the two atomic levels, $\hbar\omega$ is the photon energy, and $\lambda\hbar$ is an interaction energy. The Pauli matrices act between the two atomic levels, with $\sigma_z|\pm\rangle = \pm|\pm\rangle$, and with $\sigma_\pm = (1/2)(\sigma_x \pm i\sigma_y)$ as matrices that raise or lower the atomic energy. $\hat{a}$ and $\hat{a}^\dagger$ are the photon creation and destruction operators.

a) We introduce the notation $|+,0\rangle = |+\rangle \otimes |0\rangle$ and $|-,1\rangle = |-\rangle \otimes |1\rangle$ for the relevant product states of the composite system, with $0, 1$ referring to the photon number. Show that in the two-dimensional subspace spanned by these vectors the Hamiltonian takes the form

$$H = \frac{1}{2}\hbar\Delta \begin{pmatrix} \cos\phi & -i\sin\phi \\ +i\sin\phi & -\cos\phi \end{pmatrix} + \epsilon\mathbb{1} \tag{8}$$

where we assume $|-,1\rangle$ to correspond to the lower matrix position and $|+,0\rangle$ to the upper one. $\mathbb{1}$ denotes the $2\times 2$ identity matrix. Express the parameters $\Delta$, $\cos\phi$, $\sin\phi$, and $\epsilon$ in terms of $\omega_0$, $\omega$ and $\lambda$.

b) Find the energy eigenvalues $E_\pm$. Find also the eigenstates $|\psi_\pm(\phi)\rangle$, expressed in terms of the product states $|+,0\rangle$ and $|-,1\rangle$, and show that they are related by $|\psi_-(\phi)\rangle = |\psi_+(\phi+\pi)\rangle$.

In the following we focus on the state $|\psi_-(\phi)\rangle$, which we assume to be the one-photon state in the non-interacting case. This state (with a convenient choice of phase factor) can be written as $|\psi_-(\phi)\rangle = \cos\frac{\phi}{2}\,|-,1\rangle + i\sin\frac{\phi}{2}\,|+,0\rangle$.

c) Find expressions for the reduced density operators of the photon and of the atom for the state $|\psi_-(\phi)\rangle$. Discuss in what parameter interval the state is mostly a photon-like state and when it is mostly an atom-like state.

d) Determine the entanglement entropy as a function of $\phi$, and find for what values the entropy is minimal and maximal. Relate this to the discussion in c).

e) At time $t = 0$ a single photon is sent into the cavity, where there is previously no photon and the atom is in its ground state. Determine the time dependent probability $p(t)$ for a photon later to be present in the cavity. Give a qualitative explanation of its oscillatory behaviour, and specify what determines the frequency and amplitude of the oscillations.