**FYS 4110 Modern Quantum Mechanics, Fall Semester 2016**

## Problem set 12

### 12.1 Time evolution in a two-level system (Exam 2013)

The Hamiltonian of a two-level system (denoted $A$) is $\hat{H}_0 = (1/2)\hbar\omega\,\sigma_z$, with $\sigma_z$ as the diagonal Pauli matrix. We refer to the normalized ground state vector as $|g\rangle$ and the exited state as $|e\rangle$. In reality the system is coupled to a radiation field (denoted $S$), and the excited state will therefore decay to the ground state under emission of a quantum of radiation. $\hat{\rho}$ denotes the reduced density operator of subsystem $A$. To a good approximation the time evolution of this system is described by the Lindblad equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[H_0, \hat{\rho}\right] - \frac{1}{2}\gamma\left[\hat{\alpha}^\dagger\hat{\alpha}\hat{\rho} + \hat{\rho}\hat{\alpha}^\dagger\hat{\alpha} - 2\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger\right] \tag{1}$$

with $\gamma$ as the decay rate for the transition $|e\rangle \to |g\rangle$, $\hat{\alpha} = |g\rangle\langle e|$ and $\hat{\alpha}^\dagger = |e\rangle\langle g|$.

In matrix form, with $\{|e\rangle, |g\rangle\}$ as basis, we write the density matrix as $\hat{\rho}$

$$\hat{\rho} = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{2}$$

with $p_e$ as the probability for the system to be in state $|e\rangle$ and $p_g$ as the probability for the system to be in state $|g\rangle$.

a) Assume initially the two-level system, at time $t = 0$, to be in state $\hat{\rho} = |e\rangle\langle e|$. Show, by use of Eq. (1), that $p_e$ decays exponentially, with $\gamma$ as decay rate, while the total probability $p_e + p_g$ is conserved.

b) Assume next that the system is initially in the following superposition of the two eigenstates of $\hat{H}_0$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Determine the time dependent density matrix $\hat{\rho}(t)$ with this initial state.

c) The density operator of subsystem $A$ can alternatively be expressed in terms of the Pauli matrices as $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$. Determine the function $r^2(t)$ in the two cases above and show that in both cases it has a minimum for $t = (1/\gamma)\ln 2$. What is the minimum value for $r$ in the two cases? Comment on the implication the results give for the entanglement between the two subsystems $A$ and $S$. (We assume $A$+$S$ all the time to be in a pure state.)

### 12.2 Radiation damping (Exam 2014)

A charged particle is oscillating in a one-dimensional harmonic oscillator potential. It emits electric dipole radiation, with the rate for transition between an initial state $i$ and a final state $f$ given by the radiation formula

$$\mathcal{W}_{fi} = \frac{4\alpha}{3c^2}\omega_{fi}^3|x_{fi}|^2 \tag{3}$$

where $\alpha$ is the fine structure constant, $\hbar\omega_{fi}$ is the energy radiated in the transition, and $c$ is the speed of light. $x$ is the position coordinate of the particle, which is related to the raising and lowering operators of the harmonic oscillator by

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \tag{4}$$

with $m$ as the mass of the particle.

a) Show that the non-vanishing transition rates are of the form

$$\mathcal{W}_{n-1,n} = \gamma n \tag{5}$$

with $n = 0, 1, 2, ...$ as referring to the energy levels of the harmonic oscillator, and $\gamma$ as a constant decay parameter. Detemine $\gamma$.

The time evolution of the quantum state of the oscillating particle is described by the Lindblad equation in the following way

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[H_0, \hat{\rho}\right] - \frac{1}{2}\gamma\left[\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^\dagger\right] \tag{6}$$

with $\hat{\rho}$ as the density operator of the particle and $H_0$ as the harmonic oscillator Hamiltonian, without decay.

b) In the following we focus on the diagonal terms of the density matrix, $p_n = \rho_{nn} = \langle n|\hat{\rho}|n\rangle$, which define the occupation probabilities of the energy eigenstates. Show that they satisfy the equation

$$\frac{dp_n}{dt} = -\gamma(np_n - (n+1)p_{n+1}) \tag{7}$$

Explain why this is consistent with the expression (5) for the transition rate $\mathcal{W}_{n-1,n}$.

c) Show that Eq. (7) implies that the expectation value of the excitation energy

$$E = \langle H_0\rangle - \frac{1}{2}\hbar\omega \tag{8}$$

decays exponentially with time.