Exam FYS4110, fall semester 2016
Solutions

PROBLEM 1

a) Matrix elements of $\hat{H}$ in the two-dimensional subspace

$$\begin{aligned} \hat{H}|0,+1\rangle &= \frac{1}{2}\hbar(\omega_0+\omega_1)|0,+1\rangle + \lambda\hbar|1,-1\rangle \\ \hat{H}|1,-1\rangle &= \frac{1}{2}\hbar(3\omega_0-\omega_1)|0,+1\rangle + \lambda\hbar|0,+1\rangle \end{aligned} \tag{1}$$

In matrix form

$$H = \frac{1}{2}\hbar\begin{pmatrix} \omega_0+\omega_1 & 2\lambda \\ 2\lambda & 3\omega_0-\omega_1 \end{pmatrix} = \frac{1}{2}\hbar\Delta\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \epsilon\hbar\mathbb{1} \tag{2}$$

which gives

$$\Delta\cos\theta = \omega_1-\omega_0\,, \quad \Delta\sin\theta = 2\lambda\,, \quad \epsilon = \omega_0 \tag{3}$$

and from this

$$\Delta = \sqrt{(\omega_1-\omega_0)^2+4\lambda^2} \tag{4}$$

and

$$\cos\theta = \frac{\omega_1-\omega_0}{\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}}\,, \quad \sin\theta = \frac{2\lambda}{\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}} \tag{5}$$

b) Eigenvalue problem for the matrix

$$\begin{aligned} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \delta\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{vmatrix} \cos\theta-\delta & \sin\theta \\ \sin\theta & -\cos\theta-\delta \end{vmatrix} &= 0 \\ \Rightarrow \quad \delta^2-\cos^2\theta-\sin^2\theta &= 0 \quad \Rightarrow \quad \delta = \pm 1 \end{aligned} \tag{6}$$

Energy eigenvalues

$$E_\pm = \hbar(\epsilon \pm \frac{1}{2}\Delta) = \hbar\left(\omega_0 \pm \frac{1}{2}\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}\right) \tag{7}$$

Eigenvectors

$$(\cos\theta \mp 1)\,\alpha + \sin\theta\,\beta = 0 \quad \Rightarrow \quad \frac{\beta}{\alpha} = \pm\frac{1\mp\cos\theta}{\sin\theta} \tag{8}$$

This gives

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = N_\pm\begin{pmatrix} \pm\sin\theta \\ 1\mp\cos\theta \end{pmatrix} \tag{9}$$

with normalization factor

$$N_\pm^2 = \sin^2\theta + (1\mp\cos\theta)^2 = 2(1\mp\cos\theta) \tag{10}$$

Finally

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \frac{1}{\sqrt{2(1 \mp \cos\theta)}} \begin{pmatrix} \pm\sin\theta \\ 1 \mp \cos\theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm\sqrt{1 \pm \cos\theta} \\ \sqrt{1 \mp \cos\theta} \end{pmatrix} \tag{11}$$

and in bra-ket form

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} \left(\pm\sqrt{1 \pm \cos\theta}\, |0, +1\rangle + \sqrt{1 \mp \cos\theta}\, |1, -1\rangle \right) \tag{12}$$

c) Density operator

$$\begin{aligned} \hat{\rho}_\pm &= \frac{1}{2}(1 \pm \cos\theta)(|0\rangle\langle 0| \otimes |+1\rangle\langle +1|) + \frac{1}{2}(1 \mp \cos\theta)(|1\rangle\langle 1| \otimes |-1\rangle\langle -1|) \\ &\pm \frac{1}{2}\sin\theta(|0\rangle\langle 1| \otimes |+1\rangle\langle -1| + |1\rangle\langle 0| \otimes |-1\rangle\langle +1|) \end{aligned} \tag{13}$$

Reduced density operators

$$\begin{aligned} \text{position}: \quad & \hat{\rho}^p_\pm = \mathrm{Tr}_s\, \hat{\rho}_\pm = \frac{1}{2}(1 \pm \cos\theta)|0\rangle\langle 0| + \frac{1}{2}(1 \mp \cos\theta)|1\rangle\langle 1| \\ \text{spin}: \quad & \hat{\rho}^s_\pm = \mathrm{Tr}_p\, \hat{\rho}_\pm = \frac{1}{2}(1 \pm \cos\theta)|+1\rangle\langle +1| + \frac{1}{2}(1 \mp \cos\theta)|-1\rangle\langle -1| \end{aligned} \tag{14}$$

Entanglement entropy

$$\begin{aligned} S^p_\pm = S^s_\pm &= -[\frac{1}{2}(1 - \cos\theta)\log(\frac{1}{2}(1 - \cos\theta)) + \frac{1}{2}(1 + \cos\theta)\log(\frac{1}{2}(1 + \cos\theta)] \\ &= -[\cos^2\frac{\theta}{2}\log(\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\log(\sin^2\frac{\theta}{2})] \equiv S \end{aligned} \tag{15}$$

Maximum entanglement

$$\theta = \frac{\pi}{2}: \quad \cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2} = \frac{1}{2} \quad \Rightarrow \quad S = \log 2 \tag{16}$$

Minimum entanglement

$$\begin{aligned} \theta = 0: \quad & \cos^2\frac{\theta}{2} = 1,\ \sin^2\frac{\theta}{2} = 0 \quad \Rightarrow \quad S = 0 \\ \theta = \pi: \quad & \cos^2\frac{\theta}{2} = 0,\ \sin^2\frac{\theta}{2} = 1 \quad \Rightarrow \quad S = 0 \end{aligned} \tag{17}$$

PROBLEM 2

a) Change of variables

$$\begin{aligned} \hat{c}^\dagger\hat{c} &= \mu^2\hat{a}^\dagger\hat{a} + \nu^2\hat{b}^\dagger\hat{b} + \mu\nu(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \\ \hat{d}^\dagger\hat{d} &= \nu^2\hat{a}^\dagger\hat{a} + \mu^2\hat{b}^\dagger\hat{b} - \mu\nu(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \\ \Rightarrow \quad \omega_c\hat{c}^\dagger\hat{c} + \omega_d\hat{d}^\dagger\hat{d} &= (\mu^2\omega_c + \nu^2\omega_d)\hat{a}^\dagger\hat{a} + (\nu^2\omega_c + \mu^2\omega_d)\hat{b}^\dagger\hat{b} \\ &+ \mu\nu(\omega_c - \omega_d)(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \end{aligned} \tag{18}$$

To get the correct form for the Hamiltonian, define ω_c, ω_d, μ and ν so that the following equations are satisfied

$$\begin{array}{lrcl} \text{I} & \mu^2+\nu^2 & = & 1 \\ \text{II} & \mu^2\omega_c+\nu^2\omega_d & = & \omega \\ \text{III} & \nu^2\omega_c+\mu^2\omega_d & = & \omega \\ \text{IV} & \mu\nu(\omega_c-\omega_d) & = & \lambda \end{array} \tag{19}$$

From I, II and III follows

$$\begin{array}{lrcl} \text{IIb} & \frac{1}{2}(\omega_c+\omega_d) & = & \omega \\ \text{IIIb} & (\mu^2-\nu^2)(\omega_c-\omega_d) & = & 0 \end{array} \tag{20}$$

Since $\omega_c \neq \omega_d$ from IV, we have $\mu^2 = \nu^2 = 1/2$, and therefore (by convenient choice of sign factors) $\mu = \nu = 1/\sqrt{2}$. Inserted in IV this gives

$$\text{IVb} \qquad \frac{1}{2}(\omega_c-\omega_d) \;=\; \lambda \tag{21}$$

which together with IIb gives

$$\omega_c = \omega+\lambda\,, \quad \omega_d = \omega-\lambda \tag{22}$$

Commutation relations

$$\begin{aligned} \left[\hat{c},\hat{c}^\dagger\right] &= \mu^2\left[\hat{a},\hat{a}^\dagger\right]+\nu^2\left[\hat{b},\hat{b}^\dagger\right] = (\mu^2+\nu^2)\mathbb{1} = \mathbb{1} \\ \left[\hat{c},\hat{d}^\dagger\right] &= -\mu\nu(\left[\hat{a},\hat{a}^\dagger\right]-\left[\hat{b},\hat{b}^\dagger\right]) = 0 \end{aligned} \tag{23}$$

Similar evaluations of other commutators show that the two sets of ladder operators satify the standard commutation rules for two independent harmonic oscillators.

b) Time evolution of a coherent state

$$\begin{aligned} |\psi(t)\rangle &= \hat{\mathcal{U}}(t)|\psi(0)\rangle\,, \quad \hat{\mathcal{U}}(t) = \exp[-i(\omega_c\hat{c}^\dagger\hat{c}+\omega_d\hat{d}^\dagger\hat{d}+\omega\mathbb{1})] \\ \Rightarrow \quad \hat{c}|\psi(t)\rangle &= \hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)^{-1}\hat{c}\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\, e^{i\omega_c t\hat{c}^\dagger\hat{c}}\,\hat{c}\, e^{-i\omega_c t\hat{c}^\dagger\hat{c}}\,|\psi(0)\rangle \\ &= e^{-i\omega_c t}\hat{\mathcal{U}}(t)\,\hat{c}\,|\psi(0)\rangle \\ &= e^{-i\omega_c t}z_{c0}\,|\psi(0)\rangle \end{aligned} \tag{24}$$

$|\psi(t)\rangle$ is thus a coherent state of the c-oscillator with eigenvalue $z_c(t) = e^{-i\omega_c t}z_{c0}$. Simlar result is valid for the d- oscillator with $z_d(t) = e^{-i\omega_d t}z_{d0}$.

c) Since all the operators $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ commute, they have a common set of eigenvalues. This implies that a state which is a coherent state of $\hat{c}$, and $\hat{d}$ will also be a coherent state of $\hat{a}$ and $\hat{b}$. As follows from a) we have

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{c}-\hat{d})\,, \quad \hat{b} = \frac{1}{\sqrt{2}}(\hat{c}+\hat{d}) \tag{25}$$

The corresponding relations between the eigenvalues are

$$\begin{aligned} z_a(t) &= \frac{1}{\sqrt{2}}(z_c(t) - z_d(t)) \\ &= \frac{1}{\sqrt{2}}(e^{-i\omega_c t} z_{c0} - e^{-i\omega_d t} z_{d0}) \\ &= \frac{1}{2} e^{-i\omega t}(e^{-i\lambda t}(z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0})) \\ &= \frac{1}{2} e^{-i\omega t}(\cos(\lambda t) z_{a0} - i\sin(\lambda t) z_{b0}) \end{aligned} \tag{26}$$

and similarly

$$\begin{aligned} z_b(t) &= \frac{1}{2} e^{-i\omega t}(-e^{-i\lambda t}(z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0})) \\ &= \frac{1}{2} e^{-i\omega t}(i\sin(\lambda t) z_{a0} + \cos(\lambda t) z_{b0}) \end{aligned} \tag{27}$$

PROBLEM 3

a) Time derivatives of matrix elements

$$\begin{aligned} \text{I} \qquad \dot{p}_e &= \langle e|\frac{d\hat{\rho}}{dt}|e\rangle = -\gamma p_e + \gamma' p_g \\ \text{II} \qquad \dot{p}_g &= \langle g|\frac{d\hat{\rho}}{dt}|g\rangle = -\gamma' p_g + \gamma p_e \\ \text{III} \qquad \dot{b} &= \langle e|\frac{d\hat{\rho}}{dt}|g\rangle = [\frac{i}{\hbar}\Delta E - \frac{1}{2}(\gamma + \gamma')]b \end{aligned} \tag{28}$$

From I and II follows $\frac{d}{dt}(p_e + p_g = 0)$, the sum of occupation probabilities is constant.

b) Conditions satisfied by the density operator

$$\begin{aligned} 1) \qquad \hat{\rho} &= \hat{\rho}^\dagger \\ 2) \qquad \hat{\rho} &\geq 0 \\ 3) \qquad \mathrm{Tr}\,\hat{\rho} &= 1 \end{aligned} \tag{29}$$

1) implies that p_e and p_g are real, which is consistent with the interpretation of these as probabilities. 3) gives the normalization $p_e + p_g = 1$. 2) means that the eigenvalues of $\hat{\rho}$ are non-negative. To see the implication of this we find the eigenvalues from the secular equation

$$\begin{aligned} &\begin{vmatrix} p_e - \lambda & b \\ b^* & p_g - \lambda \end{vmatrix} = 0 \\ \Rightarrow \qquad &\lambda^2 - \lambda + p_e p_g - |b|^2 = 0 \\ \Rightarrow \qquad &\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4(|b|^2 - p_e p_g)} \end{aligned} \tag{30}$$

Positivity of λ_- then requires $|b|^2 \leq p_e p_g$.

c) At thermal equilibrium we have $\dot{p}_e = \dot{p}_g = \dot{b} = 0$. I then implies

$$\gamma p_e = \gamma' p_g \quad \Rightarrow \quad \frac{p_e}{p_g} = \frac{\gamma'}{\gamma} = e^{-\Delta E/kT} \tag{31}$$

Using $p_g = 1 - p_e$ we find

$$\begin{aligned} p_e = \frac{\gamma'/\gamma}{1+\gamma'/\gamma} &= \frac{1}{1+e^{\Delta E/kT}} \\ p_g = \frac{1}{1+\gamma'/\gamma} &= \frac{1}{1+e^{-\Delta E/kT}} \end{aligned} \tag{32}$$

From III follows $\dot{b} = 0 \Rightarrow b = 0$.

d) From the initial values $p_e(0) = 1$, $p_g(0) = 0$, and the constraint on $|b|^2$ follows

$$|b(0)|^2 \leq p_e(0)p_g(0) = 0 \quad \Rightarrow \quad b(0) = 0 \tag{33}$$

We apply in the following the general formula

$$\dot{x} = ax \quad \Rightarrow \quad x(t) = e^{at}x(0) \tag{34}$$

For b this means

$$b(t) = e^{-\frac{i}{b}\Delta E - \frac{1}{2}(\gamma+\gamma')t}\, b(0) = 0 \tag{35}$$

With $p_e = 1 - p_g$ eq. II gives for p_g

$$\dot{p}_g = -(\gamma+\gamma')p_g + \gamma = -(\gamma+\gamma')(p_g - \frac{1}{1+\gamma'/\gamma}) \tag{36}$$

or

$$\frac{d}{dt}(p_g - \frac{1}{1+\gamma'/\gamma}) = -(\gamma+\gamma')p_g + \gamma = -(\gamma+\gamma')(p_g - \frac{1}{1+\gamma'/\gamma}) \tag{37}$$

Integrating the equation gives

$$p_g(t) - \frac{1}{1+\gamma'/\gamma} = e^{-(\gamma+\gamma')t}(p_g(0) - \frac{1}{1+\gamma'/\gamma}) \tag{38}$$

which with $p_g(0) = 1$ is solved to

$$p_g(t) = \frac{1}{1+\gamma'/\gamma}(1 + (\gamma'/\gamma)e^{-(\gamma+\gamma')t}) \tag{39}$$

and for $p_e = 1 - p_g$ gives

$$p_e(t) = \frac{\gamma'/\gamma}{1+\gamma'/\gamma}(1 + e^{-(\gamma+\gamma')t}) \tag{40}$$

We note that the above expressions reproduce correctly, in the limit $t \to \infty$, the values for p_e and p_g at thermal equilibrium.

The limit $T \to 0$ gives $\gamma'/\gamma \to 0$. This gives $p_g(t) \to 1$ and $p_e(t) \to 0$ consistent with the fact that the system remains in the ground state when $T = 0$. In the limit $T \to \infty$ we have $\gamma'/\gamma \to 1$, which gives

$$\begin{aligned} p_g(t) &\to \frac{1}{2}(1 + e^{-2\gamma t}) \\ p_e(t) &\to \frac{1}{2}(1 - e^{-2\gamma t}) \end{aligned} \tag{41}$$

In this case the time evolution gives $\lim_{t\to\infty} p_e = \lim_{t\to\infty} p_g = \frac{1}{2}$.