UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	**FYS4110/9110 Modern Quantum Mechanics**
Day of exam:	**5. December 2018**
Exam hours:	**9.00-13.00, 4 hours**
This examination paper consists of 3 pages	
Permitted materials:	**Approved electronic calculator.**
	Angell and Lian: Størrelser og enheter i fysikken
	Rottmann: Matematisk formelsamling

Language: The solutions may be written in Norwegian or English depending on your own preference.
Make sure that your copy of this examination paper is complete before answering.

PROBLEM 1
Pure and mixed states

a) Explain what is the difference between pure and mixed quantum states. How are they represented mathematically?

b) An ensemble of spin-$\frac{1}{2}$ particles are produced by some (to you) unknown procedure. You are informed that the particles will be either (ensemble A) in the state $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ or (ensemble B) in a random statistical mixture with 50% of the particles in the state $|\uparrow\rangle$ and 50% of the particles in the state $|\downarrow\rangle$. You are allowed to measure the spin of each particle along an axis of your choice (you do not have to choose the same axis for each particle). Describe an experiment which would reveal whether the particles are prepared in ensemble A or ensemble B. Explain what will be the probabilities of different measurement oucomes for both ensembles when using your measurement procedure.

c) Consider now a third enemble (ensemble C), where the particles are prepared in a random statistical mixture with 50% of the particles in the state $|\rightarrow\rangle$ and 50% of the particles in the state $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. Prove that we can not distiguish ensembles B and C by any measurements on the particles.

Instead of direct preparation as described above, we can prepare the ensembles B or C remotely by entanglement in the following way. Person 1 (the preparer) prepares an ensemble of pairs of entangled particles in the state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. He keeps one particle from each pair to himself and sends the other particle from each pair to person 2 (you). By doing appropriate measurements on his particles, person 1 can now decide at a later point if he would like your particles to belong to ensemble B or C.

d) Which measurement should person 1 perform to generate ensemble B and which to generate ensemble C? Justify your answer.

e) Even if the ensembles B and C are indistiguishable by local measurements by person 2, as you showed in question c), they can be distiguished by the correlations between the measurement outcomes of persons 1 and 2. Explain which measurements person 2 should do, and how the difference between ensembles B and C are visible in the correlations. Assume that the pairs are labeled, so that we can compare the measurement oucomes for the two particles belonging to the same pair. What changes if person 1 decides to wait with his measurements until after person 2 makes the measurements, so that the two ensembles are not prepared until after they are measured.

PROBLEM 2
Coupled harmonic oscillators

Two identical harmonic oscillators, A and B, are coupled with a Hamiltonian

$$H = \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) + \hbar\lambda(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}). \tag{1}$$

Here $\hat{a}^\dagger$ and $\hat{a}$ are creation and annihilation operators for oscillator A and $\hat{b}^\dagger$ and $\hat{b}$ corresponding operators for oscillator B.

a) Show that the Hamiltonian can be expressed in diagonal form as

$$H = \hbar\omega_c\hat{c}^\dagger\hat{c} + \hbar\omega_d\hat{d}^\dagger\hat{d} \tag{2}$$

where $\hat{c}$ and $\hat{d}$ are linear combinations of $\hat{a}$ and $\hat{b}$

$$\hat{c} = \mu\hat{a} + \nu\hat{b}, \qquad \hat{d} = -\nu\hat{a} + \mu\hat{b} \tag{3}$$

where μ and ν are positive real constants satisfying $\mu^2 + \nu^2 = 1$. Determine the constants μ, ν, ω_c and ω_d in terms of ω and λ. Check that the operators $\hat{c}$ and $\hat{d}$ satisfy the usual harmonic oscillator commutation relations, and that the oscillators C and D are independent of each other (all operators for different oscillators commute).

b) We define the number operators for the original oscillators as $N_A = \hat{a}^\dagger\hat{a}$ and $N_B = \hat{b}^\dagger\hat{b}$. Assume that the initial state of the system is the first excited state of oscillator A. That is, the state $\hat{a}^\dagger|0\rangle$ where $|0\rangle$ is the ground state. Find the expectation values $\langle N_A\rangle$ and $\langle N_B\rangle$ as functions of time. Describe the result.

c) Calculate the entanglement entropy between oscillators A and B as a function of time. What is the maximal value of the entanglement entropy. At what times is the entropy zero and what is the state of the system at these times?

PROBLEM 3

Driven two-level system with damping

The Hamiltonian of an isolated two-level system is $H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$. Let $|g\rangle$ be the ground state and $|e\rangle$ the excited state. The system is coupled to a radiation field, so that the excited state spontaneously will decay to the ground state, emitting a quantum of radiation (which could be photons, phonons or some other field excitation depending on the physical realization). This means that the density matrix ρ of the system will (to a good approximation) satisfy a Lindblad equation of the form

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0, \rho] - \frac{1}{2}\gamma\left[\alpha^\dagger\alpha\rho + \rho\alpha^\dagger\alpha - 2\alpha\rho\alpha^\dagger\right] \tag{4}$$

where γ is the decay rate for the transition $|e\rangle \to |g\rangle$ and $\alpha = |g\rangle\langle e|$.

a) We parametrize the density matrix in the following way

$$\rho = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}. \tag{5}$$

Derive the equations for $\dot{p}_e$, $\dot{p}_g$ and $\dot{b}$ and check that they are consistent with the conservation of total probability, $p_e + p_g = 1$.

b) Find the solution of the Lindblad equation if the initial state is $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Calculate the Bloch vector as a function of time and describe its motion in the Bloch sphere (Reminder: The density matrix can be expressed as $\rho = \frac{1}{2}(1 + \mathbf{r}\cdot\sigma)$ where $\mathbf{r}$ is the Bloch vector).

We excite the two-level system by an external wave, which we assume is described by adding a time dependent driving term to the Hamiltonian, so that it takes the form

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \frac{1}{2}\hbar\omega_1(\cos\omega t\sigma_x + \sin\omega t\sigma_y). \tag{6}$$

c) We want to study the system in a reference frame rotating around the z-axis with the frequency ω of the external wave. That is, we define the state in the rotating frame as $|\psi'\rangle = T(t)|\psi\rangle$ where $T(t)$ is a time dependent unitary transformation. Determine the form of $T(t)$ and derive the form of the Lindblad equation in the rotating frame.

d) Find the stationary solution of the Lindblad equation in the rotating frame. Describe the result in the limiting cases of small and large ω_1. What quantity should ω_1 be compared to for the limits to apply?