FYS 4110 exam 2018 Solutions.

Problem 1

a) A pure state is the most accurate description possible of a quantum system. It is represented by a state vector $|\psi\rangle$ in Hilbert space. A mixed state is used when we do not know the exact quantum state, but ~~only~~ probabilities p_i for a set of possible states $|\psi_i\rangle$. It is represented by a density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. Mixed states also occur for composite systems in pure states. The reduced density matrix of one component is then a mixed state when there is entanglement between the component and the rest of the system.

b) We measure the spin in the x-direction. $|\rightarrow\rangle$ is an eigenstate of σ_x with eigenvalue $+1$, which means that we will measure spin up in x for all particles in ensemble A. For ensemble B we will measure spin up and spin down randomly with equal probabilities.

c) We consider the density matrices:

$$\rho_B = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow|$$

$$\rho_C = \frac{1}{2}|\rightarrow\rangle\langle\rightarrow| + \frac{1}{2}|\leftarrow\rangle\langle\leftarrow|$$

$$= \frac{1}{4}(|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + \frac{1}{4}(|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|)$$

$$= \frac{1}{4}\big(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$$

$$+ |\uparrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| \big)$$

$$= \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \rho_B$$

Since the density matrices are the same we will get the same statistics for all possible measurements, and we can not distinguish the ensembles.

d) $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

It is clear that if we measure the first particle along the z-axis we have equal probabilities of measuring up or down, and the second particle will collapse to the opposite state, generating ensemble B. Ensemble C is generated by measuring the first particle in the x-direction. To see this we rewrite $|\psi\rangle$ in terms of the states $|\rightarrow\rangle$ and $|\leftarrow\rangle$.

We have $|\uparrow\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle + |\leftarrow\rangle)$

$$|\downarrow\rangle = \frac{1}{\sqrt{2}}(|\rightarrow\rangle - |\leftarrow\rangle)$$

$$|\psi\rangle = \frac{1}{2\sqrt{2}}(|\rightarrow\rangle + |\leftarrow\rangle) \otimes (|\rightarrow\rangle - |\leftarrow\rangle) - \frac{1}{2\sqrt{2}}(|\rightarrow\rangle - |\leftarrow\rangle)(|\rightarrow\rangle + |\leftarrow\rangle)$$

$$= \frac{1}{2\sqrt{2}}\Big(|\rightarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle + |\leftarrow\rightarrow\rangle - |\leftarrow\leftarrow\rangle$$
$$- |\rightarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle + |\leftarrow\rightarrow\rangle + |\leftarrow\leftarrow\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\big(|\leftarrow\rightarrow\rangle - |\rightarrow\leftarrow\rangle\big)$$

e) Consider the case where person 1 measures spin along the z-axis and therefore prepares ensemble B. If person 2 also measures along the z-axis, the outcomes of the two measurements will always be perfectly anticorrelated. If instead person 1 measures x-spin and prepares ensemble C while person 2 still measures z-spin, the results will be uncorrelated. Nothing changes if person 1 measures after person 2.

Problem 2.

a) $H = \hbar\omega(a^+a + b^+b) + \hbar\lambda(a^+b + b^+a)$

$$H = \hbar\omega_c c^+c + \hbar\omega_d d^+d$$

$$= \hbar\omega_c(\mu a^+ + \nu b^+)(\mu a + \nu b) + \hbar\omega_d(-\nu a^+ + \mu b^+)(-\nu a + \mu b)$$

$$= \hbar\omega_c(\mu^2 a^+a + \mu\nu(a^+b + b^+a) + \nu^2 b^+b)$$

$$+ \hbar\omega_d(\nu^2 a^+a - \mu\nu(a^+b + b^+a) + \mu^2 b^+b)$$

$$= \hbar(\omega_c\mu^2 + \omega_d\nu^2)a^+a + \hbar(\omega_c\nu^2 + \omega_d\mu^2)b^+b$$

$$+ \hbar(\omega_c - \omega_d)\mu\nu(a^+b + b^+a)$$

$$\Rightarrow \left.\begin{aligned} \omega_c\mu^2 + \omega_d\nu^2 &= \omega \\ \omega_c\nu^2 + \omega_d\mu^2 &= \omega \end{aligned}\right\} \mu^2 = \nu^2 \quad \Rightarrow \mu = \nu = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{2}(\omega_c + \omega_d) = \omega$$

$$(\omega_c - \omega_d)\mu\nu = \lambda \qquad \Rightarrow \frac{1}{2}(\omega_c - \omega_d) = \lambda$$

$$\Rightarrow \omega_c = \omega + \lambda \qquad \omega_d = \omega - \lambda$$

$$[c, c^+] = [\mu a + \nu b, \mu a^+ + \nu b^+] = \mu^2 \underbrace{[a, a^+]}_{1} + \nu^2 \underbrace{[b, b^+]}_{1} = \mu^2 + \nu^2 = \underline{1}$$

$$[d, d^+] = [-\nu a + \mu b, -\nu a^+ + \mu b^+] = \nu^2[a, a^+] + \mu^2[b, b^+] = \underline{1}$$

$$[c, d] = [\mu a + \nu b, -\nu a + \mu b] = 0$$

$$[c, d^+] = [\mu a + \nu b, -\nu a^+ + \mu b^+] = -\mu\nu[a, a^+] + \mu\nu[b, b^+] = 0$$

b) $$\left.\begin{aligned} c &= \tfrac{1}{\sqrt{2}}(a+b) \\ d &= \tfrac{1}{\sqrt{2}}(-a+b) \end{aligned}\right\} \Rightarrow \begin{aligned} a &= \tfrac{1}{\sqrt{2}}(c-d) \\ b &= \tfrac{1}{\sqrt{2}}(c+d) \end{aligned}$$

$$|\psi(0)\rangle = |1_a 0_b\rangle = a^+|0\rangle = \frac{1}{\sqrt{2}}(c^+ - d^+)|0\rangle = \frac{1}{\sqrt{2}}\left(|1_c 0_d\rangle - |0_c 1_d\rangle\right)$$

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_c t c^+ c - i\omega_d t d^+ d}\left(|1_c 0_d\rangle - |0_c 1_d\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\omega_c t}|1_c 0_d\rangle - e^{-i\omega_d t}|0_c 1_d\rangle\right)$$

$$= \frac{1}{2}\left[e^{-i\omega_c t}(a^+ + b^+)|0\rangle - e^{-i\omega_d t}(-a^+ + b^+)|0\rangle\right]$$

$$= \frac{1}{2}\left[\underbrace{(e^{-i\omega_c t} + e^{-i\omega_d t})}_{A}|1_a 0_b\rangle + \underbrace{(e^{-i\omega_c t} - e^{-i\omega_d t})}_{B}|0_a 1_b\rangle\right]$$

$$\langle N_A\rangle = \langle\psi(t)|a^+ a|\psi(t)\rangle$$

$$= \frac{1}{4}(e^{i\omega_c t} + e^{i\omega_d t})(e^{-i\omega_c t} + e^{-i\omega_d t})$$

$$= \frac{1}{4}\Big(2 + \underbrace{e^{-i(\omega_c-\omega_d)t} + e^{i(\omega_c-\omega_d)t}}_{2\cos(\omega_c-\omega_d)t \,=\, 2\cos 2\lambda t}\Big)$$

$$= \frac{1}{2}(1 + \cos 2\lambda t) = \cos^2\lambda t$$

$$\langle N_B\rangle = \langle\psi(t)|b^+ b|\psi(t)\rangle$$

$$= \frac{1}{4}(e^{i\omega_c t} - e^{i\omega_d t})(e^{-i\omega_c t} - e^{-i\omega_d t})$$

$$= \frac{1}{2}(1 - \cos 2\lambda t) = \sin^2\lambda t$$

Energy is oscillating between the two oscillators.

c) $\rho_A = \mathrm{Tr}_B |\psi(t)\rangle\langle\psi(t)| = \frac{1}{4}\mathrm{Tr}_B\left(A|1_a 0_b\rangle + B|0_a 1_b\rangle\right)\left(A^*\langle 1_a 0_b| + B^*\langle 0_a 1_b|\right)$

$= \frac{1}{4}\left(|A|^2 |1_a\rangle\langle 1_a| + |B|^2 |0_a\rangle\langle 0_a|\right)$

$= \cos^2\lambda t\, |1_a\rangle\langle 1_a| + \sin^2\lambda t\, |0_a\rangle\langle 0_a|$

$S = -\cos^2\lambda t \ln\cos^2\lambda t - \sin^2\lambda t \ln \sin^2\lambda t$

Maximal value when $\cos^2\lambda t = \sin^2\lambda t = \frac{1}{2}$

$S_{max} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2$

$S = 0$ when $\cos^2\lambda t$ or $\sin^2\lambda t = 0$

$\Rightarrow \lambda t = n\frac{\pi}{2} \qquad n = 0, 1, 2, \ldots$

The system is then in state $|1_a 0_b\rangle$ or $|0_a 1_b\rangle$.

Problem 3.

a) $H_0 = \frac{1}{2}\hbar\omega_0\sigma_z$ $\qquad \rho = \begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}$ $\qquad |e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H_0,\rho] - \frac{\gamma}{2}[\alpha^+\alpha\rho + \rho\alpha^+\alpha - 2\alpha\rho\alpha^+]$$

$\alpha = |g\rangle\langle e| = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\alpha^+\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$= -\frac{i\omega_0}{2}\underbrace{\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}\right]}_{\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}}_{\begin{pmatrix} P_e & b \\ -b^* & -P_g \end{pmatrix}} - \underbrace{\begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\begin{pmatrix} P_e & -b \\ b^* & -P_g \end{pmatrix}}} - \frac{\gamma}{2}\Bigg[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}}_{\begin{pmatrix} P_e & b \\ 0 & 0 \end{pmatrix}} + \underbrace{\begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} P_e & 0 \\ b^* & 0 \end{pmatrix}} - 2\underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} P_e & b \\ b^* & P_g \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & P_e \\ 0 & b^* \end{pmatrix}}}_{\begin{pmatrix} 0 & 0 \\ 0 & P_e \end{pmatrix}}\Bigg]$$

$$= -i\omega_0\begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix} 2P_e & b \\ b^* & -2P_e \end{pmatrix}$$

$$\Rightarrow \dot{P}_e = -\gamma P_e$$

$$\dot{P}_g = \gamma P_e$$

$$\dot{b} = -\left(\frac{\gamma}{2} + i\omega_0\right)b$$

$$\frac{d}{dt}(P_e + P_g) = \dot{P}_e + \dot{P}_g = -\gamma P_e + \gamma P_e = 0$$

b) $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\rho(0) = |\psi(0)\rangle\langle\psi(0)| = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow P_e(0) = P_g(0) = b(0) = \frac{1}{2}$

$P_e(t) = e^{-\gamma t}P_e(0) = \frac{1}{2}e^{-\gamma t}$

$P_g(t) = 1 - P_e(t) = 1 - \frac{1}{2}e^{-\gamma t}$

$$b(t) = e^{-(\frac{\gamma}{2}+i\omega_0)t}\, b(0) = \frac{1}{2} e^{-(\frac{\gamma}{2}+i\omega_0)t}$$

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})_{\vec{r}=(x,y,z)} = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$\Rightarrow z = P_e - P_g = e^{-\gamma t} - 1$$

$$x = 2\,\mathrm{Re}\, b = e^{-\frac{\gamma}{2}t}\cos\omega_0 t$$

$$y = -2\,\mathrm{Im}\, b = e^{-\frac{\gamma}{2}t}\sin\omega_0 t$$

A spiral in the xy-plane starting on the surface of the Bloch sphere and decaying to the axis and a decay of the z-component to the ground state.

c) $T(t) = e^{\frac{i}{2}\omega t\sigma_z}$

$$|\psi'\rangle = T(t)|\psi\rangle$$

$$\rho' = T\rho T^{+}$$

$$\frac{d\rho'}{dt} = \dot{T}\rho T^{+} + T\rho\dot{T}^{+} + T\dot{\rho}T^{+}$$

$$= \underbrace{\frac{i}{2}\omega\sigma_z\rho' - \frac{i}{2}\omega\rho'\sigma_z}_{\frac{i}{\hbar}[\frac{\hbar}{2}\omega\sigma_z,\rho']} + T\left\{-\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}[\alpha^{+}\alpha\rho + \rho\alpha^{+}\alpha - 2\alpha\rho\alpha^{+}]\right\}T^{+}$$

$$T[H,\rho]T^{+} = TH\rho T^{+} - T\rho HT^{+} = THT^{+}\rho' - \rho' THT^{+}$$

$$T = e^{\frac{i}{2}\omega t\sigma_z} = \cos\frac{\omega t}{2} + i\sin\frac{\omega t}{2}\sigma_z$$

$$THT^{+} = \frac{1}{2}\hbar\omega_0\sigma_z + \frac{1}{2}\hbar\omega_1\left(\cos\omega t\, T\sigma_x T^{+} + \sin\omega t\, T\sigma_y T^{+}\right)$$

$$T\sigma_x T^\dagger = \left(\cos\tfrac{\omega t}{2} + i\sin\tfrac{\omega t}{2}\sigma_z\right)\sigma_x\left(\cos\tfrac{\omega t}{2} - i\sin\tfrac{\omega t}{2}\sigma_z\right)$$

$$= \cos^2\tfrac{\omega t}{2}\sigma_x + i\sin\tfrac{\omega t}{2}\cos\tfrac{\omega t}{2}\underbrace{[\sigma_z,\sigma_x]}_{2i\sigma_y} + \sin^2\tfrac{\omega t}{2}\underbrace{\sigma_z\sigma_x\sigma_z}_{-\sigma_x}$$

$$= \cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y$$

$$T\sigma_y T^\dagger = \left(\cos\tfrac{\omega t}{2} + i\sin\tfrac{\omega t}{2}\sigma_z\right)\sigma_y\left(\cos\tfrac{\omega t}{2} - i\sin\tfrac{\omega t}{2}\sigma_z\right)$$

$$= \cos^2\tfrac{\omega t}{2}\sigma_y + i\sin\tfrac{\omega t}{2}\cos\tfrac{\omega t}{2}\underbrace{[\sigma_z,\sigma_y]}_{-2i\sigma_x} + \sin^2\tfrac{\omega t}{2}\underbrace{\sigma_z\sigma_y\sigma_z}_{-\sigma_y}$$

$$= \cos\omega t\,\sigma_y + \sin\omega t\,\sigma_x$$

$$THT^\dagger = \tfrac{1}{2}\hbar\omega_0\sigma_z + \tfrac{1}{2}\hbar\omega_1\left(\cos^2\omega t\,\sigma_x - \cos\omega t\sin\omega t\,\sigma_y\right.$$
$$\left. + \cos\omega t\sin\omega t\,\sigma_y + \sin^2\omega t\,\sigma_x\right)$$

$$= \tfrac{1}{2}\hbar\omega_0\sigma_z + \tfrac{1}{2}\hbar\omega_1\sigma_x$$

$$T\alpha T^\dagger = \cos^2\tfrac{\omega t}{2}\alpha + i\sin\tfrac{\omega t}{2}\cos\tfrac{\omega t}{2}\underbrace{[\sigma_z,\alpha]}_{\underbrace{\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}}_{-\alpha} - \underbrace{\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\1&-1\end{pmatrix}}_{\alpha}} + \sin^2\tfrac{\omega t}{2}\underbrace{\sigma_z\underbrace{\alpha\sigma_z}_{\alpha}}_{-\alpha}$$

$$= (\cos\omega t - i\sin\omega t)\alpha = e^{-i\omega t}\alpha$$

$$T\alpha^\dagger T^\dagger = e^{i\omega t}\alpha^\dagger$$

$$\Rightarrow \frac{d\rho'}{dt} = -\frac{i}{\hbar}[H',\rho'] - \frac{\gamma}{2}\left[\alpha^\dagger\alpha\rho' + \rho'\alpha^\dagger\alpha - 2\alpha\rho'\alpha^\dagger\right]$$

$$H' = THT^\dagger - \tfrac{1}{2}\hbar\omega\sigma_z = \tfrac{1}{2}\hbar\underbrace{(\omega_0-\omega)}_{\Delta}\sigma_z + \tfrac{1}{2}\hbar\omega_1\sigma_x$$

d) Let $\rho' = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}$

$$[\sigma_x, \rho'] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} - \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b^*-b & p_g-p_e \\ p_e-p_g & b-b^* \end{pmatrix}$$

$$\frac{d\rho'}{dt} = -i\Delta\begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix} - \frac{i}{2}\omega_1\begin{pmatrix} b^*-b & p_g-p_e \\ p_e-p_g & b-b^* \end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix} 2p_e & b \\ b^* & -2p_e \end{pmatrix}$$

$$\dot{p}_e = -\frac{i}{2}\omega_1(b^*-b) - \gamma p_e$$

$$\dot{p}_g = \frac{i}{2}\omega_1(b^*-b) + \gamma p_e$$

$$\dot{b} = -i\Delta b - \frac{i}{2}\omega_1(p_g-p_e) - \frac{\gamma}{2}b$$

Stationary state: $\dot{p}_e = \dot{p}_g = \dot{b} = 0$

$$-\frac{i}{2}\omega_1(b^*-b) - \gamma p_e = 0$$

$$-\Delta b - \frac{i}{2}\omega_1(\underbrace{p_g}_{1-p_e}-p_e) - \frac{\gamma}{2}b = 0$$

$$\Rightarrow b = \frac{\omega_1(p_e-\frac{1}{2})}{\Delta - \frac{i\gamma}{2}} \qquad b^* = \frac{\omega_1(p_e-\frac{1}{2})}{\Delta + \frac{i\gamma}{2}}$$

$$p_e = -\frac{i\omega_1}{2\gamma}(b^*-b) = \frac{\frac{1}{4}\omega_1^2}{\Delta^2 + \frac{\gamma^2}{4} + \frac{\omega_1^2}{2}}$$

$$b = -\frac{\omega_1}{2}\,\frac{\Delta + \frac{i\gamma}{2}}{\Delta^2 + \frac{\gamma^2}{4} + \frac{\omega_1^2}{2}}$$

$\omega_1 \ll \sqrt{\Delta^2 + \frac{\gamma^2}{4}}$: $p_e \ll 1$, $|b| \ll 1$ Driving is weak and state is close to ground state.

$\omega_1 \gg \sqrt{\Delta^2 + \frac{\gamma^2}{4}}$: $p_e \approx \frac{1}{2}$ $b \approx 0$ Driving is strong and $p_e \approx p_g$. All relative phases have the same probability and $b \approx 0$.