FY110, Exam 2008

Solutions

Problem 1

a) $\hat{H}|0,+1\rangle = \frac{1}{2}\hbar(\omega_0+\omega_1)|0,+1\rangle + \lambda\hbar|1,-1\rangle$

$\hat{H}|1,-1\rangle = \frac{1}{2}\hbar(3\omega_0-\omega_1)|1,-1\rangle + \lambda\hbar|0,+1\rangle$

matrix form:

$$H = \hbar\begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \text{with} \quad a = \tfrac{1}{2}(\omega_0+\omega_1), \quad b = \lambda, \quad c = \tfrac{1}{2}(3\omega_0-\omega_1)$$

written as:

$$H = \hbar\Delta\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} + \hbar\varepsilon\mathbb{1}$$

$\Rightarrow \quad a = \Delta\cos\theta + \varepsilon, \quad b = \Delta\sin\theta, \quad c = -\Delta\cos\theta + \varepsilon$

$\Rightarrow \quad \underline{\varepsilon = \frac{1}{2}(a+b) = \omega_0}, \quad \underline{\Delta\cos\theta = \frac{1}{2}(a-b) = \frac{1}{2}(\omega_1-\omega_0)}, \quad \underline{\Delta\sin\theta = \lambda}$

b) Write $H = \hbar\Delta M + \hbar\varepsilon\mathbb{1}$

Eigenvalue problem for M: $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \delta\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

$$\Rightarrow \begin{vmatrix} \cos\theta-\delta & \sin\theta \\ \sin\theta & -\cos\theta-\delta \end{vmatrix} = 0 \Rightarrow \delta^2 - \cos^2\theta - \sin^2\theta = 0 \Rightarrow \underline{\delta = \pm 1}$$

Energy eigenvalues $\underline{E_\pm = \hbar(\varepsilon \pm \Delta)}$

Eigenvectors $(\cos\theta \mp 1)\alpha + \sin\theta\,\beta = 0 \Rightarrow \frac{\beta}{\alpha} = \mp\frac{1\pm\cos\theta}{\sin\theta}$

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix}_\pm = N_\pm\begin{pmatrix} \mp\sin\theta \\ 1\pm\cos\theta \end{pmatrix} \quad \text{with} \quad N_\pm^{-2} = \sin^2\theta + (1\pm\cos\theta)^2 = 2(1\pm\cos\theta)$$

$$\Rightarrow \begin{pmatrix}\alpha \\ \beta\end{pmatrix}_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp\frac{\sin\theta}{\sqrt{1\pm\cos\theta}} \\ \sqrt{1\pm\cos\theta} \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} \mp\sqrt{1\mp\cos\theta} \\ \sqrt{1\pm\cos\theta} \end{pmatrix}}$$

or $\underline{|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(\mp\sqrt{1\mp\cos\theta}\,|0,+1\rangle + \sqrt{1\pm\cos\theta}\,|1,-1\rangle\right)}$

c) Density operator

$$\rho_\pm = \tfrac{1}{2}(1 \mp \cos\theta)\,|0\rangle\langle 0| \otimes |{+1}\rangle\langle{+1}| + \tfrac{1}{2}(1 \pm \cos\theta)\,|1\rangle\langle 1| \otimes |{-1}\rangle\langle{-1}|$$

$$\mp \tfrac{1}{2}\sin\theta\,\big(|0\rangle\langle 1| \otimes |{+1}\rangle\langle{-1}| + |1\rangle\langle 0| \otimes |{-1}\rangle\langle{+1}|\big)$$

Reduced density operators

position $\rho^p_\pm = Tr_s\,\rho_\pm = \frac{1}{2}(1 \mp \cos\theta)\,|0\rangle\langle 0| + \frac{1}{2}(1 \pm \cos\theta)\,|1\rangle\langle 1|$

spin $\rho^s_\pm = Tr_p\,\rho_\pm = \frac{1}{2}(1 \mp \cos\theta)\,|{+1}\rangle\langle{+1}| + \frac{1}{2}(1 \pm \cos\theta)\,|{-1}\rangle\langle{-1}|$

Entropies

$$S^p_\pm = S^s_\pm = -\Big[\tfrac{1}{2}(1-\cos\theta)\log\big(\tfrac{1}{2}(1-\cos\theta)\big) + \tfrac{1}{2}(1+\cos\theta)\log\big(\tfrac{1}{2}(1+\cos\theta)\big)\Big]$$

$$= \underline{-\Big[\cos^2\tfrac{\theta}{2}\log\cos^2\tfrac{\theta}{2} + \sin^2\tfrac{\theta}{2}\log\sin^2\tfrac{\theta}{2}\Big]} \equiv S$$

gives the measure of entanglement between spin and position

$\cos\theta = 0\ (\theta = \frac{\pi}{2}) \Rightarrow \cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2} = \frac{1}{2} \Rightarrow \underline{S = \log 2}$ max. entanglement

$\cos\theta = \pm 1\ (\theta = 0, \pi) \Rightarrow \cos^2\frac{\theta}{2} = 1,\ \sin^2\frac{\theta}{2} = 0$ or $\cos^2\frac{\theta}{2} = 0,\ \sin^2\frac{\theta}{2} = 1$

$\Rightarrow \underline{S = 0}$ minimal entanglement

Problem 2

a) $x_{BA} = y_{BA} = 0$ due to rotational invariance about the z-axis (vanish under φ-integration, since ψ_A and ψ_B are φ independent)

z-component : $z = r\cos\theta \Rightarrow$

$$z_{BA} = \frac{1}{\sqrt{32}}\frac{1}{\pi a_0^3}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dr\, r^2\, r\cos\theta\, \cos\theta\, \frac{r}{a_0}\, e^{-\frac{3}{2}\frac{r}{a_0}}$$

$$= \frac{1}{4\sqrt{2}}\frac{1}{\pi}2\pi \int_0^{\pi} d\theta \sin\theta \cos^2\theta\; a_0 \int_0^{\infty}\frac{dr}{a_0}\Big(\frac{r}{a_0}\Big)^4 e^{-\frac{3}{2}\frac{r}{a_0}}$$

$$= \frac{1}{2\sqrt{2}}\Big(\frac{2}{3}\Big)^5 \int_{-1}^{1} du\, u^2 \int_0^{\infty} d\xi\, \xi^4 e^{-\xi}\, a_0 \qquad \Big(u = \cos\theta,\ \xi = \frac{3}{2}\frac{r}{a_0}\Big)$$

$$= \underline{\nu\, a_0} \qquad \nu \text{ numerical factor}$$

$$\nu = \frac{1}{2\sqrt{2}}\Big(\frac{2}{3}\Big)^5 \cdot \frac{2}{3}\cdot 4! = \underline{\frac{1}{\sqrt{2}}\frac{256}{243} = 0.745}$$

b) Probability per unit solid angle, for arbitrary polarization

$$p(\theta,\varphi) = N \sum_a |\langle B, 1_{\vec{k}a}|\hat{H}_{emis}|A,0\rangle|^2$$

$$= N' \sum_a |\vec{\varepsilon}^{\,*}_{\vec{k}a}\cdot\vec{e}_z|^2 \qquad (\vec{r}_{BA} = z_{BA}\,\vec{e}_z)$$

N, N' normalization factors

$$\sum_a |\vec{\varepsilon}^{\,*}_{\vec{k}a}\cdot\vec{e}_z|^2 = \vec{e}_z^{\,2} - \frac{(\vec{k}\cdot\vec{e}_z)^2}{k^2} = 1-\cos^2\theta = \sin^2\theta$$

Normalization of probability

$$\iint p(\theta,\varphi)\sin\theta\, d\theta\, d\varphi = 1$$

$$\Rightarrow (N')^{-1} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\,(1-\cos^2\theta)\sin\theta$$

$$= 2\pi \int_{-1}^{1} du\,(1-u^2) \qquad (u = \cos\theta)$$

$$= \frac{8\pi}{3} \quad \Rightarrow \quad \underline{p(\theta,\varphi) = \frac{3}{8\pi}\sin^2\theta}$$

c) $2s \to 1s$ is electric dipole "forbidden". Electric dipole matrix element vanishes due to selection rule for parity. Other interaction matrix elements are much smaller. Implies slower transition and longer life time.

Problem 3

a) Density operators, general properties

1) $\hat{\rho} = \hat{\rho}^+$ hermiticity
2) $\hat{\rho} \geq 0$ positivity
3 $Tr\,\hat{\rho} = 1$ normalization

Spectral decomposition (eigenvector expansion):

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k| \qquad p_k \geq 0 \quad \sum_k p_k = 1$$

Pure state: $\hat{\rho} = |\psi\rangle\langle\psi|$, only one term

Mixed state: several terms with $0 < p_k < 1$

b) Composite system, Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \quad \text{tensor product}$$

Density operator $\hat{\rho}$, acts on $\mathcal{H}$

1) Uncorrelated states, $\hat{\rho}$ factorizes

$$\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B \Rightarrow \langle \hat{A}\hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle$$

for operator $\hat{A}$ acting on $\mathcal{H}_A$ and $\hat{B}$ acting on $\mathcal{H}_B$

2) Classical correlations (separable states)

$\hat{\rho}$ expressed as a probability distribution over uncorrelated states $\hat{\rho} = \sum_{kl} \hat{\rho}^A_k \otimes \hat{\rho}^B_l \, p_{kl}$; $p_{kl} > 0$ $\sum_{kl} p_{kl} = 1$

3) Entangled states:

$\hat{\rho}$ cannot be expressed in the form 2)

Correlations in the wave functions, not simply in a probability distribution over product states.

c) Schmidt decomposition of a pure state in a composite system

$$\underline{|\psi\rangle = \sum_k c_k |k\rangle_A \otimes |k\rangle_B} \quad \text{with } \langle k|k'\rangle_A = \langle k|k'\rangle_B = \delta_{kk'}$$

any $|\psi\rangle$ can be brought into this form

Density operators $\hat{\rho} = \sum_{kk'} c_k c_{k'}^* \, |k\rangle\langle k'|_A \otimes |k\rangle\langle k'|_B$

$$\hat{\rho}_A = Tr_B \, \hat{\rho} = \sum_k |c_k|^2 |k\rangle\langle k|_A$$

$$\hat{\rho}_B = Tr_A \, \hat{\rho} = \sum_k |c_k|^2 |k\rangle\langle k|_B$$

Entropies $\underline{S_A = S_B = -\sum_k |c_k|^2 \log |c_k|^2}$

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Løsninger

Oppgave 1

a) $\hat{H}|\psi(t)\rangle = -i\hbar\lambda(\sin\lambda t\,|+-\rangle - \cos\lambda t\,|-+\rangle)$

$$= \underline{i\hbar \frac{d}{dt}|\psi(t)\rangle}$$

Tetthetsoperator

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| = \cos^2\lambda t\,|+-\rangle\langle+-| + \sin^2\lambda t\,|-+\rangle\langle-+|$$
$$\underline{+\cos\lambda t\sin\lambda t\,(|+-\rangle\langle-+| + |-+\rangle\langle+-|)}$$

b) Benytter:

$|+\rangle\langle+| = \frac{1}{2}(\mathbb{1}+\sigma_z)$, $|-\rangle\langle-| = \frac{1}{2}(\mathbb{1}-\sigma_z)$

$|+\rangle\langle-| = \sigma_+$, $|-\rangle\langle+| = \sigma_-$

$$\Rightarrow |+-\rangle\langle+-| = \frac{1}{4}(\mathbb{1}+\sigma_z)\otimes(\mathbb{1}-\sigma_z) = \frac{1}{4}(\mathbb{1} + \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$

$$|-+\rangle\langle-+| = \frac{1}{2}(\mathbb{1}-\sigma_z)\otimes(\mathbb{1}+\sigma_z) = \frac{1}{4}(\mathbb{1} - \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$

$$|+-\rangle\langle-+| = \sigma_+\otimes\sigma_- \;;\quad |-+\rangle\langle+-| = \sigma_-\otimes\sigma_+$$

$$\Rightarrow \hat{\rho}(t) = \frac{1}{4}\mathbb{1} + \frac{1}{4}(\cos^2\lambda t - \sin^2\lambda t)(\sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z) - \frac{1}{4}\sigma_z\otimes\sigma_z$$
$$+\cos\lambda t\sin\lambda t\,(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+)$$
$$\underline{= \frac{1}{4}\mathbb{1} + \frac{1}{4}\cos 2\lambda t\,(\sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z) - \frac{1}{4}\sigma_z\otimes\sigma_z + \frac{1}{2}\sin 2\lambda t\,(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+)}$$

Reduserte tetthetsoperatorer, benytter $\mathrm{Tr}\,\sigma_z = \mathrm{Tr}\,\sigma_\pm = 0$

$$\hat{\rho}_A(t) = \mathrm{Tr}_B\,\hat{\rho}(t) = \underline{\frac{1}{2}(\mathbb{1} + \cos 2\lambda t\,\sigma_z)}$$

$$\hat{\rho}_B(t) = \mathrm{Tr}_A\,\hat{\rho}(t) = \underline{\frac{1}{2}(\mathbb{1} - \cos 2\lambda t\,\sigma_z)}$$

c) Graden av sammenfiltring = von Neumann entropien til delsystemene:

$$S = -\mathrm{Tr}_A(\hat{\rho}_A \log \rho_A) = -\mathrm{Tr}_B(\hat{\rho}_B \log \hat{\rho}_B)$$

$$\hat{\rho}_A = \tfrac{1}{2}(1+\cos 2\lambda t)|+\rangle\langle+| + \tfrac{1}{2}(1-\cos 2\lambda t)|-\rangle\langle-|$$

$$= \cos^2\lambda t\,|+\rangle\langle+| + \sin^2\lambda t\,|-\rangle\langle-|$$

$$\Rightarrow \log\hat{\rho}_A = \log[\cos^2\lambda t]|+\rangle\langle+| + \log[\sin^2\lambda t]|-\rangle\langle-|$$

$$\underline{S = -(\cos^2\lambda t\log[\cos^2\lambda t] + \sin^2\lambda t\log[\sin^2\lambda t])}$$

Oppgave 2

a) $c^\dagger c = \mu^2 a^\dagger a + \nu^2 b^\dagger b + \mu\nu(a^\dagger b + b^\dagger a)$

$d^\dagger d = \nu^2 a^\dagger a + \mu^2 b^\dagger b - \mu\nu(a^\dagger b + b^\dagger a)$

$$\Rightarrow \omega_c c^\dagger c + \omega_d d^\dagger d = (\mu^2\omega_c + \nu^2\omega_d)a^\dagger a + (\nu^2\omega_c + \mu^2\omega_d)b^\dagger b + \mu\nu(\omega_c - \omega_d)(a^\dagger b + b^\dagger a)$$

Setter: $\omega = \mu^2\omega_c + \nu^2\omega_d = \nu^2\omega_c + \mu^2\omega_d$ I

og $\mu\nu(\omega_c - \omega_d) = \lambda$ II

I $\Rightarrow$ $\omega = \frac{1}{2}(\mu^2+\nu^2)(\omega_c+\omega_d) = \frac{1}{2}(\omega_c+\omega_d)$ (1)

$\Rightarrow \mu^2 = \nu^2 = \frac{1}{2}$

$\underline{\mu = \nu = \frac{1}{\sqrt{2}}}$ $\Rightarrow$ $\frac{1}{2}(\omega_c - \omega_d) = \lambda$ (2)

(1) & (2) $\Rightarrow$ $\underline{\omega_c = \omega + \lambda}$, $\underline{\omega_d = \omega - \lambda}$

Kommutasjonsrelasjoner

$[c, c^\dagger] = \mu^2[a, a^\dagger] + \nu^2[b, b^\dagger] = (\mu^2+\nu^2)\mathbb{1} = \mathbb{1}$

$[d, d^\dagger] = \nu^2[a, a^\dagger] + \mu^2[b, b^\dagger] = (\mu^2+\nu^2)\mathbf{1} = \mathbf{1}$

$[c, d^\dagger] = -\mu\nu([a, a^\dagger] - [b, b^\dagger]) = 0 \Rightarrow [c^\dagger, d] = 0$

andre kommutatorer = 0

$\Rightarrow$ To uavh. sett med harm. osc. operatorer

b) Tidsutvikling av koherent tilstand

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle \quad ; \quad \hat{U}(t) = \exp[-i(\omega_c c^\dagger c + \omega_d d^\dagger d + \omega \mathbb{1})]$$

$$\hat{c}|\psi(t)\rangle = \hat{U}(t)\hat{U}(t)^{-1}\hat{c}\,\hat{U}(t)|\psi(0)\rangle$$

$$\hat{U}(t)^{-1}\hat{c}\,\hat{U}(t) = e^{i\omega_c t\, c^\dagger c}\,\hat{c}\,e^{-i\omega t\, c^\dagger c}$$

$$= c + i\omega_c t\,[c^\dagger c, c] + \tfrac{1}{2}(i\omega_c t)^2\,[c^\dagger c,[c^\dagger c, c]] + \cdots$$

$$= (1 - i\omega_c t + \tfrac{1}{2}(-i\omega_c t)^2 + \cdots)\,c = e^{-i\omega_c t}\,c$$

$$\Rightarrow \hat{c}|\psi(t)\rangle = e^{-i\omega_c t}\,\hat{U}(t)\,\hat{c}\,|\psi(0)\rangle = \underline{e^{-i\omega_c t} z_{c0}|\psi(t)\rangle}$$

c) $\hat{c} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{b})$, $\hat{d} = -\frac{1}{\sqrt{2}}(\hat{a}-\hat{b})$

$$\Rightarrow \hat{a} = \tfrac{1}{\sqrt{2}}(\hat{c}-\hat{d}), \quad \hat{b} = \tfrac{1}{\sqrt{2}}(\hat{c}+\hat{d})$$

Operatorene har felles egentilstander med egenverdier

$$z_a(t) = \tfrac{1}{\sqrt{2}}(z_c(t) - z_d(t)) = \tfrac{1}{\sqrt{2}}(e^{-i\omega_c t} z_{c0} - e^{-i\omega_d t} z_{d0})$$

$$= \tfrac{1}{2}e^{-i\omega t}(e^{-i\lambda t}(+z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0}))$$

$$= \underline{e^{-i\omega t}(\cos\lambda t\, z_{a0} - i\sin\lambda t\, z_{b0}}$$

$$z_b(t) = -\tfrac{1}{2}e^{-i\omega t}(e^{-i\lambda t}(z_{a0} + z_{b0}) - e^{i\lambda t}(z_{a0} - z_{b0}))$$

$$= \underline{e^{-i\omega t}(i\sin\lambda t\, z_{a0} + \cos\lambda t\, z_{b0})}$$

Oppgave 3

a) Krav til tetthetsmatrise

1) Hermitisitet: $\hat{\rho}^\dagger = e^{-\beta\hat{H}^\dagger} = e^{-\beta\hat{H}} = \hat{\rho}$ (β reell)

2) Positivitet: Egenverdier $\hat{\rho}|n\rangle = e^{-\beta E_n}|n\rangle$

$e^{-\beta E_n} > 0$ for alle n

3) Normering $\mathrm{Tr}\,\hat{\rho} = 1 \Leftrightarrow N^{-1} = \mathrm{Tr}\, e^{-\beta\hat{H}}$

bestemmer N

Normeringskonstant

$$N^{-1} = \sum_n e^{-\beta E_n} = e^{-\frac{1}{2}\beta\hbar\omega} \sum_{n=0}^{\infty} (e^{-\beta\hbar\omega})^n = \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1\cancel{2}}{2\sinh(\frac{1}{2}\beta\hbar\omega)}$$

$$\underline{N = 2\cancel{\tfrac{1}{2}}\sinh(\tfrac{1}{2}\beta\hbar\omega)}$$

b) Forventningsverdi for energien

$$E = Tr(N e^{-\beta\hat{H}}\hat{H}) = -N\frac{d}{d\beta} Tr(e^{-\beta\hat{H}})$$

$$= -N\frac{d}{d\beta}(N^{-1}) = \underline{\frac{1}{N}\frac{dN}{d\beta}}$$

$$\frac{dN}{d\beta} = \frac{1}{4}\hbar\omega\cosh(\tfrac{1}{2}\beta\hbar\omega) \Rightarrow E = \tfrac{1}{2}\hbar\omega\coth(\tfrac{1}{2}\beta\hbar\omega)$$

$\beta \to \infty$: $\coth(\frac{1}{2}\beta\hbar\omega) \to 1 \Rightarrow \underline{E \to \frac{1}{2}\hbar\omega}$ grunntilst. energien

c) $$\hat{\rho} = \int \frac{d^2z}{\pi} p(|z|)\,|z\rangle\langle z| = \sum_{n,n'} \underbrace{\int \frac{d^2z}{\pi} p(|z|)\,\langle n|z\rangle\langle z|n'\rangle}_{\equiv I_{nn'}}\, |n\rangle\langle n'|$$

$$I_{nn'} = \int \frac{d^2z}{\pi} p(|z|)\frac{z^n z^{*n'}}{\sqrt{n!\,n'!}} e^{-|z|^2}$$

$$= \frac{1}{\pi}\int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r p(r) \frac{r^{n+n'} e^{i\varphi(n-n')}}{\sqrt{n!\,n'!}} e^{-r^2} \; ; \quad \int_0^{2\pi} e^{i\varphi(n-n')} d\varphi = 2\pi\delta_{nn'}$$

$$= 2\int_0^{r} dr\, r^{2n+1} e^{-r^2} p(r)\frac{1}{n!}\delta_{nn'}$$

$$\Rightarrow \hat{\rho} = \sum_n p_n |n\rangle\langle n| \quad \text{med} \quad \underline{p_n = \frac{2}{n!}\int_0^{\infty} dr\, r^{2n+1} e^{-r^2} p(r)}$$

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Løsninger

Oppgave 1

a) En tilstandsvektor eller tetthetsoperator som ikke er på tensor-produktform inneholder korrelasjoner mellom delsystemene.
Her er det en ren tilstand som ikke er på produktform,
$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$.
Korrelasjonene ligger i tilstandsvektoren, ikke i tetthetsoperatoren, dvs $\hat{\rho} = |\psi\rangle\langle\psi| \neq \sum_k p_k \hat{\rho}_k^A \otimes \hat{\rho}_k^B \otimes \hat{\rho}_k^C$; tilstanden er ikke separabel, men sammenfiltret.

b) Tetthetsoperator

$$\hat{\rho} = \tfrac{1}{2}(|uuu\rangle\langle uuu| + |ddd\rangle\langle ddd| - |uuu\rangle\langle ddd| - |ddd\rangle\langle uuu|)$$

Reduserte tetthetsoperatorer

$$\hat{\rho}_A = \mathrm{Tr}_{BC}\,\hat{\rho} = \tfrac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|)_A = \tfrac{1}{2}\mathbb{1}_A$$

$$\hat{\rho}_{BC} = \mathrm{Tr}_A\,\hat{\rho} = \tfrac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd|)_{BC}$$

Sammenfiltringsentropien til et todelt system som er i en ren kvante-tilstand, er lik von Neumann-entropien til delsystemene (som er like).

Her $S = S_A = S_{BC} = -\sum_k p_k \log p_k = -2(\tfrac{1}{2}\log\tfrac{1}{2}) = \underline{\log 2}$

$\hat{\rho}_A$ er maksimalt blandet, dvs S_A har maksimal verdi

$\Rightarrow$ S maksimal, de to delsystemene er maksimalt sammenfiltret.

Delsystem BC: $\hat{\rho}_{BC} = \tfrac{1}{2}(\hat{\rho}_u^B \otimes \hat{\rho}_u^C + \hat{\rho}_d^B \otimes \hat{\rho}_d^C)$; $\hat{\rho}_u = |u\rangle\langle u|$, $\hat{\rho}_d = |d\rangle\langle d|$

$\hat{\rho}_{BC}$ separabel $\Rightarrow$ B og C <u>ikke</u> sammenfiltret.

c) Uttrykker $|\psi\rangle$ ved $|f\rangle$ og $|b\rangle$ for delsystem A

$$|u\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) \; ; \; |d\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle) \quad \Rightarrow$$

$$|\psi\rangle = \frac{1}{2}\left(|f\rangle \otimes (|uu\rangle - |dd\rangle) + |b\rangle \otimes (|uu\rangle + |dd\rangle)\right)$$

Måling med f som resultat $\Rightarrow$ ny kvantetilstand proporsjonal med $|f\rangle_A$ $\Rightarrow$ $|\psi\rangle' = \frac{1}{\sqrt{2}}|f\rangle \otimes (|uu\rangle - |dd\rangle)$ etter måling

$$\equiv |\psi_A'\rangle \otimes |\psi_{BC}'\rangle$$

Tetthetsoperator

$$\hat{\rho}' = |\psi'\rangle\langle\psi'| = |\psi_A'\rangle\langle\psi_A'| \otimes |\psi_{BC}'\rangle\langle\psi_{BC}'| \equiv \hat{\rho}_A' \otimes \hat{\rho}_{BC}'$$

Delsystemene A og BC ikke lenger korrelerte

$$\hat{\rho}_{BC}' = \frac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd| - |uu\rangle\langle dd| - |dd\rangle\langle uu|)$$

$$\Rightarrow \hat{\rho}_B' = \mathrm{Tr}_C\, \hat{\rho}_{BC}' = \frac{1}{2}\mathbb{1}_B \; ; \; \hat{\rho}_C' = \frac{1}{2}\mathbb{1}_C$$

Delsystemet BC er i en ren tilstand; undersystemene B og C er maksimalt blandet; dvs. spinntilstanden for B og C er maksimalt sammenfiltret.

a) Vinkelavhengigheten til matriseelementet sitter i faktoren $(\vec{k} \times \vec{\varepsilon}_{ka}) \cdot \vec{\sigma}_{BA} = \vec{\varepsilon}_{ka} \cdot (\vec{\sigma}_{BA} \times \vec{k})$. Sannsynlighetsfordelingen $p(\theta,\varphi)$ er uavhengig av polarisasjonen, då vi summerer over a,

$$p(\theta,\varphi) = N \sum_a |\vec{\varepsilon}_{ka} \cdot (\vec{\sigma}_{BA} \times \vec{k})|^2$$

$$= N\, |\vec{\sigma}_{BA} \times \vec{k}|^2 \qquad \frac{\vec{k}}{k} \cdot (\vec{\sigma}_{BA} \times \vec{k}) = 0$$

N: normeringsfaktor bestemt av $\int d\varphi \int d\theta \sin\theta\, p(\theta,\varphi) = 1$

$$\vec{\sigma}_{BA} = (0\;1)\begin{pmatrix} \vec{e}_z & \vec{e}_x - i\vec{e}_y \\ \vec{e}_x + i\vec{e}_y & -\vec{e}_z \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{e}_x + i\vec{e}_y$$

$$\vec{k} = k(\sin\theta\cos\varphi\, \vec{e}_x + \sin\theta\sin\varphi\, \vec{e}_y + \cos\theta\, \vec{e}_z)$$

$$\Rightarrow \vec{\sigma}_{BA} \times \vec{k} = k(i\cos\theta\, \vec{e}_x - \cos\theta\, e_y - i\sin\theta\, e^{i\varphi}\vec{e}_z)$$

$$\Rightarrow |\vec{\sigma}_{BA} \times \vec{k}|^2 = k^2(2\cos^2\theta + \sin^2\theta) = k^2(1 + \cos^2\theta) \quad \text{uavh. av } \varphi$$

$$p(\theta,\varphi) = N k^2 (1+\cos^2\theta)$$

$$\Rightarrow \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \, p(\theta,\varphi) = 2\pi N k^2 \int_{-1}^{1} du (1+u^2) \qquad u = -\cos\theta$$

$$= 2\pi N k^2 \left[u + \tfrac{1}{3} u^3 \right]_{-1}^{1} = \tfrac{16}{3}\pi N k^2$$

normering: $N = \frac{3}{16\pi} \frac{1}{k^2}$

$$\Rightarrow \underline{p(\theta,\varphi) = \frac{3}{16\pi}(1+\cos^2\theta)}$$

b) $\vec{k} = k\vec{e}_x \Rightarrow$

$$|\vec{\varepsilon}(\alpha)\cdot(\vec{\sigma}_{BA}\times\vec{k})|^2 = k^2 |(\cos\alpha\,\vec{e}_y + \sin\alpha\,\vec{e}_z)\cdot(-i\vec{e}_z)|^2$$
$$= k^2 \sin^2\alpha$$

Sannsynlighetsfordeling

$$p(\alpha) = N' \sin^2\alpha$$

$$\int_0^{\pi} p(\alpha)\,d\alpha = N' \int_0^{\pi} \sin^2\alpha \, d\alpha = N' \frac{\pi}{2}$$

(definerer $0 \le \alpha < \pi$, siden α og $\alpha+\pi$ def. samme polarisasjonstilstand)

Normering $\Rightarrow N' = \frac{2}{\pi} \Rightarrow \underline{p(\alpha) = \frac{2}{\pi}\sin^2\alpha}$

c) $P_A(t) = e^{-t/\tau_A} = 1 - \frac{t}{\tau_A} + \cdots$

for små t ($t \ll \tau_A$): $P_A \simeq 1 - (\frac{1}{\tau_A})t$

Overgangssannsynlighet pr. tid for $A \to B$: $w_{BA} = \frac{1}{\tau_A}$

$$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2 \frac{e^2\hbar^3}{8Vm^2\omega\varepsilon_0} \sum_\alpha |(\vec{k}\times\vec{\varepsilon}_\alpha)\cdot\vec{\sigma}_{BA}|^2 \delta(\omega-\omega_0) \qquad k = \omega/c$$

$$= \frac{e^2\hbar\omega_0}{32\pi^2 m^2 \varepsilon_0 c^2} \frac{\omega_0^2}{c^3} \frac{16\pi}{3} \underbrace{\int d\varphi \int d\theta \sin\theta\, p(\theta,\varphi)}_{=1}$$

$$= \frac{1}{6\pi} \frac{e^2\hbar\omega_0^3}{m^2\varepsilon_0 c^5}$$

$$\Rightarrow \underline{\tau_A = 6\pi \frac{m^2\varepsilon_0 c^5}{e^2\hbar\omega_0^3}}$$

Oppgave 3

a) $$\frac{d\hat{a}}{dt} = \frac{i}{\hbar}[\hat{H},\hat{a}] = -i\omega_0\hat{a} - i\lambda e^{-i\omega t}\mathbb{1} \equiv \dot{\hat{a}}$$

$$\frac{d^2\hat{a}}{dt^2} = \frac{i}{\hbar}[\hat{H},\dot{\hat{a}}] + \frac{\partial}{\partial t}\dot{\hat{a}} = -i\omega_0(-i\omega_0\hat{a} - i\lambda e^{-i\omega t}\mathbb{1}) - i\lambda(-i\omega)e^{-i\omega t}\mathbb{1}$$

$$= -\omega_0^2\hat{a} - \lambda(\omega_0+\omega)e^{-i\omega t}\mathbb{1}$$

$$\hat{x} = \frac{1}{2}(\hat{a}+\hat{a}^+) \Rightarrow$$

$$\underline{\frac{d^2\hat{x}}{dt^2} + \omega_0^2\hat{x} = -\lambda(\omega_0+\omega)\cos\omega t} \qquad \underline{C = -\lambda(\omega_0+\omega)}$$

b) $$i\hbar\frac{d}{dt}|\psi_T(t)\rangle = \hat{T}(t)\hat{H}(t)|\psi(t)\rangle + i\hbar\frac{d\hat{T}}{dt}|\psi(t)\rangle$$

$$= \hat{H}_T(t)|\psi_T(t)\rangle$$

hvor $\hat{H}_T(t) = \hat{T}(t)\hat{H}(t)\hat{T}^+(t) + i\hbar\frac{d\hat{T}}{dt}\hat{T}^+(t)$

$$\hat{T}\hat{a}\hat{T}^+ = e^{i\omega t\hat{a}^+\hat{a}}\,\hat{a}\,e^{-i\omega t\hat{a}^+\hat{a}} = \hat{a}e^{-i\omega t} \qquad \hat{T}\hat{a}^+\hat{T}^+ = \hat{a}^+e^{i\omega t}$$

$$\Rightarrow \hat{T}\hat{H}\hat{T} = \hbar\omega_0(\hat{a}^+\hat{a}+\tfrac{1}{2}) + \hbar\lambda(\hat{a}^++\hat{a})$$

$$i\hbar\frac{d\hat{T}}{dt}\hat{T}^+ = -\hbar\omega\hat{a}^+\hat{a}$$

$$\Rightarrow \underline{\hat{H}_T = \hbar(\omega_0-\omega)\hat{a}^+\hat{a} + \hbar\lambda(\hat{a}+\hat{a}^+) + \tfrac{1}{2}\hbar\omega_0\mathbb{1}}$$

c) $|\psi_T(t)\rangle = \hat{U}_T(t)|\psi_T(0)\rangle$, $\hat{U}_T(t) = e^{-\frac{i}{\hbar}\hat{H}_T t}$

$\Rightarrow |\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$, $\hat{U}(t) = \hat{T}^+(t)\hat{U}_T(t) = e^{-i\omega t\hat{a}^+\hat{a}}e^{-\frac{i}{\hbar}\hat{H}_T t}$

Antar $|\psi(0)\rangle = |0\rangle$, $\hat{a}|0\rangle = 0$

Sjekker om $|\psi(t)\rangle$ er en koherent tilstand ved å anvende $\hat{a}$,

$$\hat{a}|\psi(t)\rangle = \hat{U}(t)\hat{U}^+(t)\hat{a}\hat{U}(t)|\psi(0)\rangle$$

$$\hat{U}^+(t)\hat{a}\hat{U}(t) = e^{\frac{i}{\hbar}\hat{H}_T t}e^{+i\omega t\hat{a}^+\hat{a}}\,\hat{a}\,e^{-i\omega t\hat{a}^+\hat{a}}e^{-\frac{i}{\hbar}\hat{H}_T t}$$

$$= e^{\frac{i}{\hbar}\hat{H}t}e^{-i\omega t}\hat{a}\,e^{-\frac{i}{\hbar}\hat{H}t}$$

$$[\hat{H}_T, \hat{a}] = \hbar(\omega-\omega_0)\hat{a} - \hbar\lambda\mathbb{1}$$

$$[\hat{H}_T, [\hat{H}_T, \hat{a}]] = \hbar(\omega-\omega_0)(\hbar(\omega-\omega_0)\hat{a} - \hbar\lambda\mathbb{1})$$

....

$$\Rightarrow e^{\frac{i}{\hbar}\hat{H}_T t}\,\hat{a}\,e^{-\frac{i}{\hbar}\hat{H}_T t} = \hat{a} + \frac{i}{\hbar}[\hat{H}_T, \hat{a}] + \frac{1}{2!}\left(\frac{i}{\hbar}\right)^2[\hat{H}_T, [\hat{H}_T, \hat{a}]] + \cdots$$

$$= \left(1 + i(\omega-\omega_0)t + \frac{1}{2!}[i(\omega-\omega_0)t]^2 + \cdots\right)\hat{a}$$

$$-i\lambda\left(t + \frac{1}{2!}(i(\omega-\omega_0))\,t^2 + \cdots \right)\mathbb{1}$$

$$= e^{i(\omega-\omega_0)t}\hat{a} - \frac{\lambda}{\omega-\omega_0}\left(e^{i(\omega-\omega_0)t} - 1\right)\mathbb{1}$$

$$\Rightarrow \hat{a}\,\hat{U}(t) = \hat{U}(t)\left(e^{-i\omega_0 t}\hat{a} - \frac{\lambda}{\omega-\omega_0}\left(e^{-i\omega_0 t} - e^{-i\omega t}\right)\mathbb{1}\right)$$

$$\Rightarrow \hat{a}\,|\psi(t)\rangle = \underline{-\frac{\lambda}{\omega-\omega_0}\left(e^{-i\omega_0 t} - e^{-i\omega t}\right)|\psi(t)\rangle}$$

egentilstand for $\hat{a}$, med egenverdi

$$\underline{z(t) = -\frac{\lambda}{\omega-\omega_0}\left(e^{-i\omega_0 t} - e^{-i\omega t}\right)}$$

Bevegelsesligning

$$\ddot{z} = -\frac{\lambda}{\omega-\omega_0}\left(-\omega_0^2 e^{-i\omega_0 t} + \omega^2 e^{-i\omega t}\right)$$

$$= -\omega_0^2 z - \frac{\lambda}{\omega-\omega_0}\left(\omega^2-\omega_0^2\right)e^{-i\omega t}$$

$$\ddot{z} + \omega_0^2 z = -\lambda(\omega+\omega_0)e^{-i\omega t}$$

Realdel $\underline{\ddot{x} + \omega_0^2 z = -\lambda(\omega+\omega_0)\cos\omega t}$ som for $\hat{x}$

Bevegelse i z-planet: Spiralerende bane med $|z| = 0$ når $e^{-i\omega_0 t}$ og $e^{-i\omega t}$ er i motfase og $|z| = \frac{2\lambda}{|\omega-\omega_0|}$ (maksimal) når — " — er i fase.

FYS4110/9110, Exam 2011

Solutions

Problem 1

a) Matrix elements of the Hamiltonian

$$\hat{H}|-,1\rangle = (-\tfrac{1}{2}\hbar\omega_0 + \hbar\omega)|-,1\rangle - i\hbar\lambda|+,0\rangle$$

$$\hat{H}|+,0\rangle = \tfrac{1}{2}\hbar\omega_0|+,0\rangle + i\hbar\lambda|-,1\rangle$$

$$\Rightarrow \langle -,1|\hat{H}|-,1\rangle = \tfrac{1}{2}\hbar(2\omega-\omega_0)$$

$$\langle +,0|\hat{H}|+,0\rangle = \tfrac{1}{2}\hbar\omega_0$$

$$\langle -,1|\hat{H}|+,0\rangle = i\hbar\lambda$$

$$\langle +,0|\hat{H}|-,1\rangle = -i\hbar\lambda$$

in matrix form

$$H = \tfrac{1}{2}\hbar\begin{pmatrix}\omega_0 & -2i\lambda \\ 2i\lambda & 2\omega-\omega_0\end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix}\omega_0-\omega & -2i\lambda \\ 2i\lambda & \omega-\omega_0\end{pmatrix} + \tfrac{1}{2}\hbar\omega\,\mathbb{1}$$

$$= \tfrac{1}{2}\hbar\Delta\begin{pmatrix}\cos\varphi & -i\sin\varphi \\ i\sin\varphi & -\cos\varphi\end{pmatrix} + \varepsilon\,\mathbb{1}$$

with $\Delta\cos\varphi = \omega_0-\omega$, $\Delta\sin\varphi = 2\lambda$, $\underline{\varepsilon = \tfrac{1}{2}\hbar\omega}$

$$\Rightarrow \underline{\Delta = \sqrt{(\omega-\omega_0)^2+4\lambda^2}}, \quad \underline{\cos\varphi = \frac{\omega_0-\omega}{\Delta}}, \quad \underline{\sin\varphi = \frac{2\lambda}{\Delta}}$$

b) Eigenvectors determined by

$$\begin{pmatrix}\cos\varphi & -i\sin\varphi \\ i\sin\varphi & -\cos\varphi\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \mu\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

$$\begin{vmatrix}\cos\varphi-\mu & -i\sin\varphi \\ i\sin\varphi & -\cos\varphi-\mu\end{vmatrix} = 0 \quad \Rightarrow \quad \mu = \pm 1$$

Energies $E_\pm = \tfrac{1}{2}\hbar\omega \pm \tfrac{1}{2}\hbar\Delta = \underline{\tfrac{1}{2}\hbar(\omega \pm \sqrt{(\omega-\omega_0)^2+4\lambda^2})}$

Eigenvectors

$$\cos\varphi\, \alpha_\pm - i \sin\varphi\, \beta_\pm = \pm \alpha_\pm$$

$$(\cos\varphi \mp 1)\, \alpha_\pm - i \sin\varphi\, \beta_\pm = 0$$

$$\Rightarrow \quad \alpha_\pm = N i \sin\varphi \,, \quad \beta_\pm = N(\cos\varphi \mp 1)$$

normalization $N^2(\sin^2\varphi + (\cos\varphi \mp 1)^2) = 1$

$$\Rightarrow \quad N = \frac{1}{\sqrt{2(1 \mp \cos\varphi)}}$$

$$\psi_\pm(\varphi) = \frac{1}{\sqrt{2(1\mp\cos\varphi)}} \begin{pmatrix} i\sin\varphi \\ \cos\varphi \mp 1 \end{pmatrix}$$

$$\sin\varphi = 2\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2} \;;\; \cos\varphi = 2\cos^2\tfrac{\varphi}{2} - 1 = 1 - 2\sin^2\tfrac{\varphi}{2}$$

$$\Rightarrow |\psi_+(\varphi)\rangle = -\sin\tfrac{\varphi}{2}\, |-,1\rangle + i\cos\tfrac{\varphi}{2}\, |+,0\rangle$$

$$|\psi_-(\varphi)\rangle = \cos\tfrac{\varphi}{2}\, |-,1\rangle + i \sin\tfrac{\varphi}{2}\, |+,0\rangle$$

$$\cos\left(\frac{\varphi+\pi}{2}\right) = -\sin\frac{\varphi}{2} \,, \quad \sin\left(\frac{\varphi+\pi}{2}\right) = \cos\frac{\varphi}{2}$$

$$\Rightarrow |\psi_-(\varphi+\pi)\rangle = |\psi_+(\varphi)\rangle$$

c) Density operator of the $|\psi_-(\varphi)\rangle$ state

$$\rho(\varphi) = |\psi_-(\varphi)\rangle\langle\psi_-(\varphi)|$$

$$= \cos^2\tfrac{\varphi}{2} |-,1\rangle\langle -,1| + \sin^2\tfrac{\varphi}{2} |+,0\rangle\langle +,0| + i\cos\tfrac{\varphi}{2}\sin\tfrac{\varphi}{2}\left(|+,0\rangle\langle -,1| - |-,1\rangle\langle +,0|\right)$$

$$\rho_{ph}(\varphi) = \langle -|\rho(\varphi)|-\rangle + \langle +|\rho(\varphi)|+\rangle = \sin^2\tfrac{\varphi}{2}\, |0\rangle\langle 0| + \cos^2\tfrac{\varphi}{2}\, |1\rangle\langle 1|$$

$$\rho_{atom}(\varphi) = \langle 0|\rho(\varphi)|0\rangle + \langle 1|\rho(\varphi)|1\rangle = \cos^2\tfrac{\varphi}{2}\, |-\rangle\langle -| + \sin^2\tfrac{\varphi}{2}\, |+\rangle\langle +|$$

$\cos^2\frac{\varphi}{2} > \sin^2\frac{\varphi}{2}$ $\left(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\right)$: the state is mainly a one-photon state

$\cos^2\frac{\varphi}{2} < \sin^2\frac{\varphi}{2}$ $\left(\frac{\pi}{2} < \varphi < 3\frac{\pi}{2}\right)$: the state is mainly an excited atomic state

d) Entanglement entropy

$$S = -\mathrm{Tr}_{ph}(\rho_{ph}\log\rho_{ph}) = -\mathrm{Tr}_{atom}(\rho_{atom}\log\rho_{atom})$$

$$= -\left(\cos^2\tfrac{\varphi}{2}\log(\cos^2\tfrac{\varphi}{2}) + \sin^2\tfrac{\varphi}{2}\log(\sin^2\tfrac{\varphi}{2})\right)$$

Min. value when $|\psi_-(\varphi)\rangle$ is a product state:

$\cos\frac{\varphi}{2} = 0$ or $\sin\frac{\varphi}{2} = 0 \Rightarrow \varphi = 0, \pi$

gives $\underline{S=0}$

Max. value, when ρ_{ph} (ρ_{atom}) is maximally mixed:

$\cos^2\frac{\varphi}{2} = \sin^2\frac{\varphi}{2} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{2}, 3\frac{\pi}{2}$

$\Rightarrow \rho_{ph} = \frac{1}{2}\mathbb{1} \Rightarrow \underline{S = \log 2}$ max. entangled

e) Time evolution: expand in energy eigenstates

$$|\psi(0)\rangle = |-,1\rangle = \cos\frac{\varphi}{2}|\psi_-(\varphi)\rangle - \sin\frac{\varphi}{2}|\psi_+(\varphi)\rangle$$

$$\Rightarrow |\psi(t)\rangle = \cos\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_-t}|\psi_-(\varphi)\rangle - \sin\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_+t}|\psi_+(\varphi)\rangle$$

$$= \left(\cos^2\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_-t} + \sin^2\frac{\varphi}{2}e^{-\frac{i}{\hbar}E_+t}\right)|-,1\rangle$$

$$+ i\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\left(e^{-\frac{i}{\hbar}E_-t} - e^{-\frac{i}{\hbar}E_+t}\right)|+,0\rangle$$

Probability for a photon present

$$p(t) = |\langle -,1|\psi(t)\rangle|^2 = \cos^4\frac{\varphi}{2} + \sin^4\frac{\varphi}{2}$$

$$+ \cos^2\frac{\varphi}{2}\sin^2\frac{\varphi}{2}\left(e^{-\frac{i}{\hbar}(E_--E_+)t} + e^{+\frac{i}{\hbar}(E_--E_+)t}\right)$$

$$= \frac{1}{4}(1+\cos\varphi)^2 + \frac{1}{4}(1-\cos\varphi)^2 + \frac{1}{2}\sin^2\varphi\cos\left(\frac{E_--E_+}{\hbar}t\right)$$

$$= \underline{\frac{1}{2}\left(1+\cos^2\varphi + \sin^2\varphi\cos\Delta t\right)} \qquad \Delta = \sqrt{(\omega-\omega_0)^2 + 4\lambda^2}$$

Oscillations due to time dependent mixing of the one-photon state with the excited atom state. Frequency $\underline{\Delta}$, amplitude $\frac{1}{2}\sin^2\varphi = \underline{\frac{2\lambda^2}{(\omega-\omega_0)^2+4\lambda^2}}$

Problem 2

a) Time evolution of the two-level system, $\kappa = 0$:

$$U(t) = e^{-\frac{i}{2}\omega_A t \sigma_z} = \begin{pmatrix} e^{-\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{\frac{i}{2}\omega_A t} \end{pmatrix}$$

$$\rho_A(t) = U(t)\rho_A(0)U^\dagger(t)$$

$$= \begin{pmatrix} e^{-\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{\frac{i}{2}\omega_A t} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\omega_A t} & 0 \\ 0 & e^{-\frac{i}{2}\omega_A t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+z & e^{-i\omega_A t}(x-iy) \\ e^{i\omega_A t}(x+iy) & 1-z \end{pmatrix} \Rightarrow x(t)+iy(t) = e^{i\omega_A t}(x+iy)$$

$$\Rightarrow \quad x(t) = x\cos\omega_A t - y\sin\omega_A t$$
$$y(t) = x\sin\omega_A t + y\cos\omega_A t$$
$$z(t) = z$$

Precession of $\vec{r}$ around the z-axis, with ang. freq. ω_A

b) Interaction matrix element

$$\langle -, 1_k | \hat{H}_{int} | +, 0\rangle = \kappa\sqrt{\frac{\hbar}{2L\omega_k}}$$

decay rate:

$$\gamma = \frac{L}{(2\pi\hbar)^2}\int dk \frac{\kappa\hbar^2}{2L\omega_k}\delta(\omega_k - \omega_A) \qquad k = \frac{\omega_k}{c}$$

$$= \frac{L}{4\pi^2\hbar^2}\frac{\kappa^2\hbar}{2Lc\omega_A} = \underline{\frac{\kappa^2}{8\pi^2\hbar c\,\omega_A}}$$

c) $|\psi(t)\rangle = |\phi(t)\rangle \otimes |0\rangle + \sum_k c_k(t)|-, 1_k\rangle$

with $|\phi(t)\rangle = e^{-\frac{i}{2}\omega_A t - \gamma t/2}\alpha|+\rangle + e^{\frac{i}{2}\omega_A t}\beta|-\rangle$

Normalization

$$\langle\psi(t)|\psi(t)\rangle = \langle\phi(t)|\phi(t)\rangle + \sum_k |c_k(t)|^2$$

$$= e^{-\gamma t}|\alpha|^2 + |\beta|^2 + \sum_k |c_k(t)|^2 \overset{!}{=} 1$$

$$\Rightarrow \sum_k |c_k(t)|^2 = \underline{|\alpha|^2(1-e^{-\gamma t})}$$

Reduced density operator of the two-level system

$$\rho_A(t) = \mathrm{Tr}_B(|\psi(t)\rangle\langle\psi(t)|) = |\phi(t)\rangle\langle\phi(t)| + \sum_k |c_k(t)|^2 |-\rangle\langle-|$$

$$= e^{-\gamma t}|\alpha|^2|+\rangle\langle+| + (1-e^{-\gamma t}|\alpha|^2)|-\rangle\langle-|$$

$$\underline{+ e^{-\gamma t/2}(\alpha\beta^* e^{-i\omega_A t}|+\rangle\langle-| + \alpha^*\beta e^{i\omega_A t}|-\rangle\langle+|)}$$

d) $\alpha = 1$, $\beta = 0$:

$$\rho_A(t) = e^{-\gamma t}|+\rangle\langle+| + (1-e^{-\gamma t})|-\rangle\langle-|$$

$$= \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1-e^{-\gamma t} \end{pmatrix}$$

$$\Rightarrow z(t) = \underline{2e^{-\gamma t}-1}, \quad \underline{x(t) = y(t) = 0}$$

The excited state decays exponentially into the ground state, as expected

$t=0$ and $t\to\infty$ ($z=\pm1$) pure product state, $S_A = 0$

Intermediate time: $e^{-\gamma t} = \frac{1}{2} \Rightarrow \rho_A = \frac{1}{2}\mathbb{1}$, maximally entangled

e) $\alpha = \beta = \frac{1}{\sqrt{2}}$:

$$\rho_A(t) = \tfrac{1}{2} e^{-\gamma t} |+\rangle\langle +| + (1 - \tfrac{1}{2} e^{-\gamma t}) |-\rangle\langle -|$$
$$+ \tfrac{1}{2} e^{-\gamma t/2} \left(e^{-i\omega_A t} |+\rangle\langle -| + e^{i\omega_A t} |-\rangle\langle +| \right)$$
$$= \frac{1}{2} \begin{pmatrix} e^{-\gamma t} & e^{-\gamma t/2} e^{-i\omega_A t} \\ e^{-\gamma t/2} e^{i\omega_A t} & 2 - e^{-\gamma t} \end{pmatrix} \Rightarrow x(t) + i y(t) = e^{-\gamma t/2} e^{i\omega_A t}$$

$\underline{x(t) = e^{-\gamma t/2} \cos\omega_A t}$, $\underline{y(t) = e^{-\gamma t/2} \sin\omega_A t}$; $\underline{z(t) = e^{-\gamma t} - 1}$

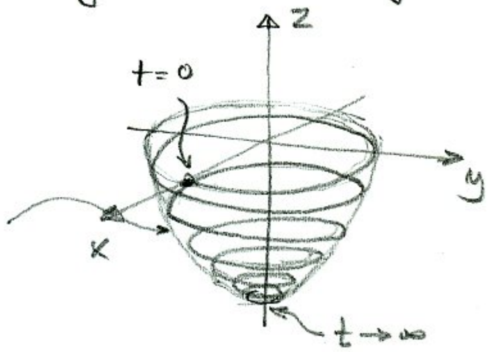
Combination of motions in a) and d) :

$\gamma \ll \omega_A \Rightarrow$ rapid precession of $\vec{r}$ around the z-axis, combined with slow decay towards the ground state

Sketch of the motion

$x^2 + y^2 = z + 1$

$\Rightarrow$ parabolic surface



$r^2 = e^{-\gamma t} + (e^{-\gamma t} + 1)^2$

$\quad = \underline{1 - e^{-\gamma t} + e^{-2\gamma t}}$

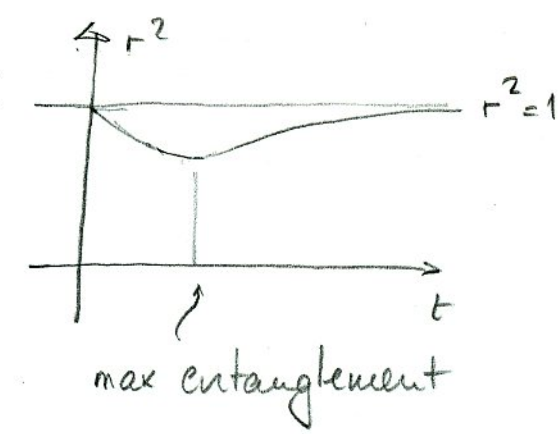
$t = 0 : r^2 = 1$, $t \to \infty : r^2 \to 1$ ent. entropy $S_A = 0$

Intermediate times $0 < r^2 < 1$

min value for $e^{-\gamma t} = \frac{1}{2}$

$\Rightarrow r^2 = \frac{3}{4}$

gives max value for S_A



FYS4110 Eksamensoppgaver 2012

Løsninger

Problem 1

a) Hamiltonian applied to the product states

$$\hat{H}|++\rangle = \tfrac{1}{2}\hbar(\omega_1+\omega_2)|++\rangle$$

$$\hat{H}|--\rangle = -\tfrac{1}{2}\hbar(\omega_1+\omega_2)|--\rangle$$

$$\hat{H}|+-\rangle = \tfrac{1}{2}\hbar\Delta|+-\rangle + \tfrac{1}{2}\hbar\lambda|-+\rangle$$

$$\hat{H}|-+\rangle = -\tfrac{1}{2}\hbar\Delta|-+\rangle + \tfrac{1}{2}\hbar\lambda|+-\rangle$$

In the subspace spanned by $|+-\rangle$ and $|-+\rangle$,

$$H = \tfrac{1}{2}\hbar\begin{pmatrix}\Delta & \lambda \\ \lambda & -\Delta\end{pmatrix} = \tfrac{1}{2}\hbar\mu\begin{pmatrix}\cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi\end{pmatrix}$$

The matrix is determined by φ, with μ as a scale factor. This implies that the eigenstates are determined by φ.

b) Eigenvalues in subspace

$$\begin{vmatrix}\cos\varphi-\varepsilon & \sin\varphi \\ \sin\varphi & -\cos\varphi-\varepsilon\end{vmatrix} = 0 \Rightarrow \varepsilon_\pm = \pm 1 .$$

energies $E_\pm = \pm\tfrac{1}{2}\hbar\mu = \underline{\pm\tfrac{1}{2}\hbar\sqrt{\Delta^2+\lambda^2}}$

Eigenstates

$$(\cos\varphi \mp 1)\alpha_\pm + \sin\varphi\,\beta_\pm = 0$$

$$(\cos\varphi \pm 1)\beta_\pm - \sin\varphi\,\alpha_\pm = 0$$

$$\Rightarrow (\cos\varphi \mp 1)\beta_\mp - \sin\varphi\,\alpha_\mp = 0$$

$$\frac{\beta_+}{\alpha_+} = -\frac{\alpha_-}{\beta_-} = -\frac{\cos\varphi - 1}{\sin\varphi} = \frac{2\sin^2\varphi/2}{2\cos\varphi/2\,\sin\varphi/2} = \tan\varphi/2$$

Normalized solutions

$$\alpha_+ = \cos\tfrac{\varphi}{2} \quad \beta_+ = \sin\tfrac{\varphi}{2} \qquad |\psi_+\rangle = \cos\tfrac{\varphi}{2}|+-\rangle + \sin\tfrac{\varphi}{2}|-+\rangle$$

$$\alpha_- = \sin\tfrac{\varphi}{2} \quad \beta_- = -\cos\tfrac{\varphi}{2} \qquad |\psi_-\rangle = \sin\tfrac{\varphi}{2}|+-\rangle - \cos\tfrac{\varphi}{2}|-+\rangle$$

c) $\Delta = 0 \Rightarrow \cos\varphi = 0 \Rightarrow \varphi = \pi/2 \Rightarrow \cos\frac{\varphi}{2} = \sin\frac{\varphi}{2} = \frac{1}{\sqrt{2}}$

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|+-\rangle \pm |-+\rangle)$$

$$|\pm\mp\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle \pm |\psi_-\rangle) = |\psi(0)\rangle$$

Time evolution

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{2}\mu t}|\psi_+\rangle + e^{\frac{i}{2}\mu t}|\psi_-\rangle\right) \qquad \mu = \lambda$$

$$= \frac{1}{2}\left(e^{-\frac{i}{2}\mu t}(|+-\rangle + |-+\rangle) + e^{\frac{i}{2}\mu t}(|+-\rangle - |-+\rangle)\right)$$

$$= \underline{\cos(\tfrac{\mu t}{2})|+-\rangle - i\sin(\tfrac{\mu t}{2})|-+\rangle} \equiv c(t)|+-\rangle + i s(t)|-+\rangle$$

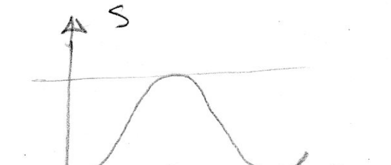
Density operator

$$\rho(t) = c(t)^2|+-\rangle\langle+-| + s(t)^2|-+\rangle\langle-+|$$
$$+ c(t)s(t)\left(|+-\rangle\langle-+| + |-+\rangle\langle+-|\right)$$

Reduced density operators

$$\rho_1(t) = c(t)^2|+\rangle\langle+| + s(t)^2|-\rangle\langle-|$$

$$\rho_2(t) = c(t)^2|-\rangle\langle-| + s(t)^2|+\rangle\langle+|$$



Entanglement entropy

$$S_1 = S_2 = -c^2\log c^2 - s^2\log s^2$$

max value : $c^2 = s^2 = \frac{1}{2} \Rightarrow S_A = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = \log 2$

min value : $c^2 = 1 \vee s^2 = 1$ $\quad S = 0$ for $c = 0 \vee s = 0$, $t = 0, \frac{\pi}{\mu}, \frac{\pi}{2\mu}, \ldots$

period $\underline{T = \frac{\pi}{\mu}}$

Problem 2

a) Hamiltonian applied to the product states

$$\hat{H}|g,1\rangle = \hbar(\tfrac{1}{2}\omega - i\gamma)|g,1\rangle + \tfrac{1}{2}\hbar\lambda|e,0\rangle$$

$$\hat{H}|e,0\rangle = \tfrac{1}{2}\hbar\omega|e,0\rangle + \tfrac{1}{2}\hbar\lambda|g,1\rangle$$

$$\hat{H}|g,0\rangle = -\tfrac{1}{2}\hbar\omega|g,0\rangle$$

In the space spanned by $|g,1\rangle$ and $|e,0\rangle$

$$H = \tfrac{1}{2}\hbar(\omega - i\gamma)\mathbb{I} + \tfrac{1}{2}\hbar\begin{pmatrix} -i\gamma & \lambda \\ \lambda & i\gamma \end{pmatrix} \equiv H_0 + H_1$$

b) Define $|\psi(t)\rangle = e^{-\frac{i}{2}\omega t - \frac{1}{2}\gamma t}|\phi(t)\rangle$

$$|\phi(t)\rangle = (\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle + ib\sin(\Omega t)|g,1\rangle$$

$\Rightarrow \quad |\psi(0)\rangle = |\phi(0)\rangle = |e,0\rangle$

satisfies the initial condition

need to show that $|\psi(t)\rangle$ satisfies the Schrödinger eq.

Note $\quad i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \quad \mathrm{I}$

$\Leftrightarrow \quad i\hbar\frac{d}{dt}|\phi(t)\rangle = \hat{H}_1|\phi(t)\rangle \quad \mathrm{II}$

Need to show that II is satisfied

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = i\hbar\Omega\left[ib\cos(\Omega t)|g,1\rangle + (-\sin\Omega t + a\cos(\Omega t))|e,0\rangle\right]$$

$$\hat{H}_1|\phi(t)\rangle = \tfrac{1}{2}\hbar\{\gamma b\sin(\Omega t) + \lambda(\cos(\Omega t) + a\sin(\Omega t))\}|g,1\rangle$$

$$\tfrac{1}{2}\hbar(i\lambda b\sin\Omega t + i\gamma(\cos(\Omega t) + a\sin(\Omega t))|e,0\rangle$$

$$= \tfrac{1}{2}\hbar\left[\{\lambda\cos(\Omega t) + (a\lambda + \gamma b)\sin(\Omega t)\}|g,1\rangle\right.$$
$$\left. + i\{\gamma\cos\Omega t + (\lambda b + \gamma a)\sin(\Omega t)\}|e,0\rangle\right]$$

Conditions for equality

$$-\Omega b = \tfrac{1}{2}\lambda \qquad \mathrm{I}$$

$$a\lambda + \gamma b = 0 \qquad \mathrm{II}$$

$$\Omega a = \tfrac{1}{2}\gamma \qquad \mathrm{III}$$

$$-\Omega = \tfrac{1}{2}(\lambda b + \gamma a) \qquad \mathrm{IV}$$

$$\mathrm{I} \Rightarrow \quad \underline{b = -\frac{\lambda}{2\Omega}} \qquad \mathrm{III} \quad \underline{a = \frac{\gamma}{2\Omega}}$$

$$\Rightarrow a\lambda + \gamma b = \frac{\gamma\lambda - \gamma\lambda}{2\Omega} = 0$$ consistent with II

$$\mathrm{IV} \Rightarrow \Omega = \frac{1}{4\Omega}(\lambda^2 - \gamma^2)$$

$$\Omega^2 = \tfrac{1}{4}(\lambda^2 - \gamma^2) \Rightarrow \underline{\Omega = \tfrac{1}{2}\sqrt{\gamma^2 - \lambda^2}}$$

c) Assume $Tr\,\rho_{tot} = 1$

$$\Rightarrow Tr\rho(t) + f(t) = 1 \qquad f(t) = 1 - Tr\rho(t)$$

$$Tr\rho(t) = \langle\psi(t)|\psi(t)\rangle = e^{-\gamma t}\langle\phi(t)|\phi(t)\rangle$$

$$\langle\phi(t)|\phi(t)\rangle = \cos^2(\Omega t) + a^2\sin^2(\Omega t) + 2a\cos\Omega t\sin\Omega t + b^2\sin^2\Omega t$$

$$= 1 + (a^2+b^2-1)\sin^2\Omega t + 2a\cos\Omega t\sin\Omega t$$

$$= 1 + \tfrac{1}{2}(a^2+b^2-1) - \tfrac{1}{2}(a^2+b^2-1)\cos(2\Omega t) + a\sin(2\Omega t)$$

$$a^2+b^2-1 = \frac{\lambda^2+\gamma^2}{\lambda^2-\gamma^2} - 1 = \frac{2\gamma^2}{\lambda^2-\gamma^2}$$

$$1 + \tfrac{1}{2}(a^2+b^2-1) = 1 + \frac{\gamma^2}{\lambda^2-\gamma^2} = \frac{\lambda^2}{\lambda^2-\gamma^2}$$

$$= Tr\rho = \underline{e^{-\gamma t}\left(\frac{\lambda^2}{\lambda^2-\gamma^2} - \frac{\gamma^2}{\lambda^2-\gamma^2}\cos(\sqrt{\lambda^2-\gamma^2}\,t) + \frac{\gamma}{\sqrt{\lambda^2-\gamma^2}}\sin(\sqrt{\lambda^2-\gamma^2}\,t)\right)}$$

$$\underline{f(t) = 1 - Tr\rho(t)}$$

The decay of $\mathrm{Tr}\rho$ is due to the leakage of the cavity photon out of the system. For the cavity states this corresponds to the transition $|g,1\rangle \to |g,0\rangle$. The second term in Eq. (5) gives the build up of probability in $|g,0\rangle$ due to this process.

With $\gamma = 0$, there are oscillations between $|g,1\rangle$ and $|e,0\rangle$ due to the coupling between the atom and the cavity photon. The time evolution of the state (4) shows, for $\gamma \neq 0$, decay of the probabilities due to the leakage $|g,1\rangle \to |g,0\rangle$, superimposed on these oscillations.

Problem 3

a) The full density operator

$$\rho_n = \tfrac{1}{3}\{ |+--\rangle\langle+--| + |-+-\rangle\langle-+-| + |--+\rangle\langle--+|$$
$$+ \eta^n (|-+-\rangle\langle+--| + |+--\rangle\langle--+|) + (\eta^*)^n (|+--\rangle\langle-+-| + |--+\rangle\langle+--|)$$
$$+ \eta^{2n} |-+-\rangle\langle--+| + (\eta^*)^{2n} |--+\rangle\langle-+-|$$

Reduced density operator

$$\rho_n^A = \mathrm{Tr}_{BC}\,\rho_n = \tfrac{1}{3}(|+\rangle\langle+| + 2|-\rangle\langle-|)$$ independent of n,

information about n can therefore not be detected by A

Measurement by A, B, C in basis I, gives result determined by probabilities of the form $\langle abc|\rho_n|abc\rangle$ with $|abc\rangle$ as a product of states $|\pm\rangle$. Only the diagonal terms in ρ_n give contributions, and these are independent of n.

Again there are no measurable differences between different n.

b) Reduced density operator

$$\rho_n^{AB} = \mathrm{Tr}_C \rho_n = \frac{1}{3}\{|+-\rangle\langle+-| + |-+\rangle\langle-+| + |--\rangle\langle--|$$
$$+ \eta^n |-+\rangle\langle+-| + (\eta^*)^n |+-\rangle\langle-+|\}$$

probabilities $p(k|n) = \langle\phi_k|\rho_n^{AB}|\phi_k\rangle$

Need overlap between vectors of basis I and II:

$$\langle 0|+\rangle = \langle 0|-\rangle = \langle 1|+\rangle = \frac{1}{\sqrt{2}} \quad \langle 1|-\rangle = -\frac{1}{\sqrt{2}}$$

note: only sign change for $\langle 1|-\rangle$

$$p(1|0) = \langle 00|\rho_0^{AB}|00\rangle = \frac{1}{3}\,\frac{5}{4} = \frac{5}{12}$$
$$p(2|0) = \langle 01|\rho_0^{AB}|01\rangle = \frac{1}{3}\left(\frac{3}{4} - \frac{2}{4}\right) = \frac{1}{12}$$
$$p(1|1) = \langle 00|\rho_1^{AB}|00\rangle = \frac{1}{3}\left(\frac{3}{4} + \frac{\eta+\eta^*}{4}\right) = \frac{1}{6}$$
$$p(2|1) = \langle 01|\rho_1^{AB}|01\rangle = \frac{1}{3}\left(\frac{3}{4} - \frac{\eta+\eta^*}{4}\right) = \frac{1}{3}$$

Have used $\eta + \eta^* = -1$

The change $n=1 \to n=2$ corresponds to $\eta \to \eta^*$ since $\eta^2 = \eta^*$

no change since the probabilities are real

c) Normalization of probabilities

$$\sum_n \bar{p}(n|k) = 1 \Rightarrow p(k) = \sum_n p(k|n)$$

$$p(1) = p(1|0) + p(1|1) + p(1|2) = \frac{5}{12} + 2\cdot\frac{1}{6} = \frac{9}{12} = \frac{3}{4}$$

Probabilities for $k=1$, $n = 0, 1, 2$

$$\bar{p}(0|1) = \frac{p(1|0)}{p(1)} = \frac{5}{12}\cdot\frac{12}{9} = \frac{5}{9}$$
$$\bar{p}(1|1) = \frac{p(1|1)}{p(1)} = \frac{1}{6}\cdot\frac{12}{9} = \frac{2}{9}$$
$$\bar{p}(2|1) = \text{—} \quad \text{"} \quad \text{—} = \frac{2}{9}$$

The message $n=0$ is most probable, with probability $\frac{5}{9}$, while $n=1$ and 2 have probability $\frac{2}{9}$.

FYS 4110 / 9110 Eksamen 2013

Løsninger

Oppgave 1

a) Utnytter $\hat{\alpha}^\dagger\hat{\alpha} = |e\rangle\langle g|g\rangle\langle e| = |e\rangle\langle e|$

Lindbladligning

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}] - \frac{1}{2}\gamma\{|e\rangle\langle e|\hat{\rho} + \hat{\rho}|e\rangle\langle e| - 2|g\rangle\langle e|\hat{\rho}|e\rangle\langle g|\}$$

for matriseelementer, utnytter

$$\langle e|[\hat{H}_0, \hat{\rho}]|e\rangle = \langle g|[\hat{H}_0, \hat{\rho}]|g\rangle = 0$$

$$\langle e|[\hat{H}_0, \hat{\rho}]|g\rangle = (E_e - E_g)\langle e|\hat{\rho}|g\rangle = \hbar\omega\langle e|\hat{\rho}|g\rangle$$

$$\Rightarrow \quad \frac{dp_e}{dt} = -\gamma\, p_e \qquad p_e(t) = e^{-\gamma t}p_e(0)$$

$$\frac{dp_g}{dt} = \gamma\, p_e \quad \Rightarrow \quad p_g(t) = 1 - p_e(t)$$

$$\frac{db}{dt} = (-i\omega - \tfrac{1}{2}\gamma)\, b \quad \Rightarrow \quad b(t) = e^{-i\omega t - \frac{1}{2}\gamma t}\, b(0)$$

Initialbetingelser

$$p_e(0) = 1, \quad p_g(0) = 0, \quad b(0) = 0$$

$$\Rightarrow \quad \underline{p_e(t) = e^{-\gamma t}}, \quad \underline{p_g(t) = 1 - e^{-\gamma t}}, \quad \underline{b(t) = 0}$$

b) Nye initialbetingelser

$$p_e(0) = |\langle e|\psi\rangle|^2 = \tfrac{1}{2}$$

$$p_g(0) = |\langle g|\psi\rangle|^2 = \tfrac{1}{2}$$

$$b(0) = \langle e|\psi\rangle\langle\psi|g\rangle = \tfrac{1}{2}$$

Tidsutvikling

$$p_e(t) = \tfrac{1}{2} e^{-\gamma t}, \; p_g(t) = 1 - \tfrac{1}{2} e^{-\gamma t}, \; b(t) = \tfrac{1}{2} e^{-i\omega t - \frac{1}{2}\gamma t}$$

$$\Rightarrow \quad \rho(t) = \frac{1}{2} \begin{pmatrix} e^{-\gamma t} & e^{-i\omega t - \frac{1}{2}\gamma t} \\ e^{i\omega t - \frac{1}{2}\gamma t} & 2 - e^{-\gamma t} \end{pmatrix}$$

c)

$$\hat{\rho} = \tfrac{1}{2}(\mathbb{1} + \vec{r}.\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}$$

$$\Rightarrow z = p_e - p_g, \; x = 2\,\mathrm{Re}\,b, \; y = -2\,\mathrm{Im}\,b$$

$$\Rightarrow r^2 = (p_e - p_g)^2 + 4|b|^2$$

Tilfelle a):

$$r^2 = (2e^{-\gamma t} - 1)^2$$

minimum for $e^{-\gamma t} = \frac{1}{2}$, $\underline{t = \frac{1}{\gamma}\ln 2}$ $\quad \underline{r_{min} = 0}$

$\Rightarrow \hat{\rho} = \frac{1}{2}\mathbb{1}$, maksimalt blandet $\Rightarrow$ A+B er maksimalt sammenfiltret.

Tilfelle b)

$$r^2 = (e^{-\gamma t} - 1)^2 + e^{-\gamma t} = e^{-2\gamma t} - e^{-\gamma t} + 1$$

$\frac{d}{dt} r^2 = 0 \Rightarrow -2e^{-2\gamma t} + e^{-\gamma t} = 0 \Rightarrow e^{-\gamma t} = \frac{1}{2}$, $\underline{t = \frac{1}{\gamma}\ln 2}$

$\Rightarrow r^2_{min} = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$, $\quad \underline{r_{min} = \frac{1}{2}\sqrt{3}}$

Siden $r_{min} < 1$ er $\hat{\rho}$ en blandet tilstand,

$\Rightarrow$ A+B er sammenfiltret, men mindre enn i tilfellet a)

I begge tilfeller er $r = 1$ både for $t = 0$ og $t \to \infty$, dvs. sammenfiltringen er bare midlertidig under henfallet $|\psi\rangle_{init} \to |g\rangle$.

Oppgave 2

a) Reduserte tetthetsoperatorer

$$\hat{\rho}_A = \mathrm{Tr}_{BC}(|\psi\rangle\langle\psi|) = \tfrac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|) = \underline{\tfrac{1}{2}\mathbb{1}_A}$$

$$\hat{\rho}_{BC} = \mathrm{Tr}_A(|\psi\rangle\langle\psi|) = \underline{\tfrac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd|)}$$

$\hat{\rho}_A$ er maksimalt blandet $\Rightarrow$ sammenfiltringsentropien er maksimal: $S = -\mathrm{Tr}_A(\hat{\rho}_A \log \rho_A) = \log 2$

$\hat{\rho}_{BC}$ er separabel, dvs en sum av produkttilstander,

$|u\rangle \otimes |u\rangle$ og $|d\rangle \otimes |d\rangle$. Ingen sammenfiltring

b) Uttrykker A-tilstanden i $|\frac{\pi}{2}, +\rangle \equiv |f\rangle$ og $|\frac{\pi}{2}, -\rangle \equiv |b\rangle$

$|u\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle)$, $|d\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle)$

$$\Rightarrow |\psi\rangle = \tfrac{1}{2}|f\rangle \otimes (|uu\rangle + |dd\rangle) + \tfrac{1}{2}|b\rangle \otimes (|uu\rangle - |dd\rangle)$$

Målingen gir f (spinn opp) $\Rightarrow$

$|\psi\rangle \to |\psi'\rangle = \frac{1}{\sqrt{2}}|f\rangle \otimes (|uu\rangle + |dd\rangle)$ normert

$$\hat{\rho}_{BC} \to \hat{\rho}'_{BC} = \tfrac{1}{2}(|uu\rangle\langle uu| + |dd\rangle\langle dd| + |uu\rangle\langle dd| + |dd\rangle\langle uu|)$$

Dette er en ren tilstand

$$\hat{\rho}_B = \mathrm{Tr}_C\, \hat{\rho}_{BC} = \tfrac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|) = \tfrac{1}{2}\mathbb{1}_B$$

Denne er maksimalt blandet $\Rightarrow$ B+C er maks. sammenfiltret

Målingen på A gjør B+C sammenfiltret!

c) Roterte tilstander

$$|u\rangle = \cos\tfrac{\theta}{2}\,|\theta,+\rangle - \sin\tfrac{\theta}{2}\,|\theta,-\rangle$$

$$|d\rangle = \sin\tfrac{\theta}{2}\,|\theta,+\rangle + \cos\tfrac{\theta}{2}\,|\theta,-\rangle$$

$\Longrightarrow$

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\{|\theta,+\rangle \otimes (\cos\tfrac{\theta}{2}|uu\rangle + \sin\tfrac{\theta}{2}|dd\rangle)$$

$$+ \;|\theta,-\rangle \otimes (-\sin\tfrac{\theta}{2}|uu\rangle + \cos\tfrac{\theta}{2}|dd\rangle)\}$$

Måleresultat $(\theta,+)$ $\Rightarrow$

$$|\psi\rangle \to |\theta,+\rangle \otimes (\cos\tfrac{\theta}{2}|uu\rangle + \sin\tfrac{\theta}{2}|dd\rangle)$$

$$\equiv |\theta,+\rangle \otimes |\psi'_{BC}(\theta)\rangle$$

$\hat{\rho}_{BC} \to \hat{\rho}'_{BC} = |\psi'_{BC}\rangle\langle\psi'_{BC}|$ ren tilstand

$$= \cos^2\tfrac{\theta}{2}\,|uu\rangle\langle uu| + \sin^2\tfrac{\theta}{2}\,|dd\rangle\langle dd|$$

$$+ \cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\,(|uu\rangle\langle dd| + |dd\rangle\langle uu|)$$

Redusert tetthetsoperator

$$\hat{\rho}_B = \mathrm{Tr}_C\,\hat{\rho}_{BC} = \cos^2\tfrac{\theta}{2}\,|u\rangle\langle u| + \sin^2\tfrac{\theta}{2}\,|d\rangle\langle d|$$

$\langle u|d\rangle = 0 \Rightarrow \cos^2\frac{\theta}{2}$ og $\sin^2\frac{\theta}{2}$ er egenverdier til $\hat{\rho}_B$

Entropi $S = -\cos^2\frac{\theta}{2}\ln(\cos^2\frac{\theta}{2}) - \sin^2\frac{\theta}{2}\ln\sin^2\frac{\theta}{2}$

= sammenfiltringsentropien mellom B og C

Oppgave 3

a) $\vec{\sigma} = \sigma_x \vec{e}_x + \sigma_y \vec{e}_y + \sigma_z \vec{e}_z$

$$= \begin{pmatrix} \vec{e}_z & \vec{e}_x - i\vec{e}_y \\ \vec{e}_x + i\vec{e}_y & -\vec{e}_z \end{pmatrix}$$

$$\vec{\sigma}_{BA} = (0\ 1)\left(- - - \right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{e}_x + i\vec{e}_y \equiv \vec{e}_+$$

$$(\vec{k} \times \vec{\varepsilon}_{ka}) \cdot \vec{e}_+ = (\vec{e}_+ \times \vec{k}) \cdot \vec{\varepsilon}_{ka}$$

$$\vec{k} = k(\cos\varphi \sin\theta \, \vec{e}_x + \sin\varphi \sin\theta \, \vec{e}_y + \cos\theta \, \vec{e}_z)$$

$$\Rightarrow \vec{e}_+ \times \vec{k} = ik(\cos\theta \, \vec{e}_+ - e^{i\varphi} \sin\theta \, \vec{e}_z)$$

Vinkelavhengighet til $|\langle B 1_{ka} | \hat{H}_1 | A, 0 \rangle|^2$:

$$p(\theta, \varphi) = N \sum_a |(\vec{e}_+ \times \vec{k}) \cdot \vec{\varepsilon}_{ka}|^2 \qquad = 0 \qquad N \text{ norm. faktor}$$

$$= N\left(|\vec{e}_+ \times \vec{k}|^2 - |(\vec{e}_+ \times \vec{k}) \cdot \frac{\vec{k}}{k}|^2\right)$$

$$= Nk^2(2\cos^2\theta + \sin^2\theta) \qquad |\vec{e}_+|^2 = 2$$

$$= Nk^2(1 + \cos^2\theta) \qquad \text{uavh av } \varphi$$

Normering $\int d\varphi \int d\theta \sin\theta (1 + \cos^2\theta) = 2\pi \int_{-1}^{1} (1 + u^2) \, du = 2\pi \left[u + \frac{1}{3}u^3\right]_{-1}^{1}$

$$= \frac{16}{3}\pi$$

$$\Rightarrow \underline{p(\theta, \varphi) = \frac{3}{16\pi}(1 + \cos^2\theta)}$$

b) $\vec{k} = k\vec{e}_x$

Sannsynlighet for deteksjon av foton med polarisasjon i retning $\vec{\varepsilon}(\alpha)$, $\qquad \vec{e}_+ \times \vec{e}_x = -i\vec{e}_z$

$$p(\alpha) = N' |(\vec{e}_+ \times \vec{e}_x) \cdot \vec{\varepsilon}(\alpha)|^2$$

$$= N' |\vec{e}_z \cdot \vec{\varepsilon}(\vec{\alpha})|^2$$

$$= N' \sin^2\alpha$$

$$p(\alpha) + p(\alpha + \tfrac{\pi}{2}) = N' = 1 \quad \Rightarrow \quad \underline{p(\alpha) = \sin^2\alpha}$$

Sannsynlighet for deteksjon:

$p(0) = 0 \qquad \alpha = 0 \Rightarrow \vec{\varepsilon} = \vec{e}_y$

$p(\frac{\pi}{2}) = 1 \qquad \alpha = \frac{\pi}{2} \Rightarrow \vec{\varepsilon} = \vec{e}_z$

viser at fotoner utsendt langs x-aksen
er polarisert langs z-aksen

FYS4110, Exam 2014

Solutions

Problem 1

a) $$\hat{\rho}_I = \cos^2 x \, |1\rangle\langle 1| + \sin^2 x \, |2\rangle\langle 2| + \cos x \sin x \, (|1\rangle\langle 2| + |2\rangle\langle 1|)$$

$$= \tfrac{1}{2}\cos^2 x \, (|+-\rangle\langle +-| + |-+\rangle\langle -+| + |+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$+ \tfrac{1}{2}\sin^2 x \, (|+-\rangle\langle +-| + |-+\rangle\langle -+| - |+-\rangle\langle -+| - |-+\rangle\langle +-|)$$

$$+ \tfrac{1}{2}\cos x \sin x \, (|+-\rangle\langle +-| - |-+\rangle\langle -+| - |+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$+ \tfrac{1}{2}\cos x \sin x \, (|+-\rangle\langle +-| - |-+\rangle\langle -+| + |+-\rangle\langle -+| - |-+\rangle\langle +-|)$$

$$= \underline{\tfrac{1}{2}(1+\sin(2x))|+-\rangle\langle +-| + \tfrac{1}{2}(1-\sin(2x))|-+\rangle\langle -+|}$$

$$\underline{+ \tfrac{1}{2}\cos 2x \, (|+-\rangle\langle -+| + |-+\rangle\langle +-|)}$$

Reduced density operators

$$\hat{\rho}_{IA} = \mathrm{Tr}_B \, \hat{\rho}_I = \tfrac{1}{2}(1+\sin(2x))|+\rangle\langle +| + \tfrac{1}{2}(1-\sin(2x))|-\rangle\langle -|$$

$$= \underline{\tfrac{1}{2}(\mathbb{1} + \sin(2x)\,\sigma_z)}$$

$$\hat{\rho}_{IB} = \mathrm{Tr}_A \, \hat{\rho}_I = \tfrac{1}{2}(1-\sin(2x))|+\rangle\langle +| + \tfrac{1}{2}(1+\sin(2x))|-\rangle\langle -|$$

$$= \underline{\tfrac{1}{2}(\mathbb{1} - \sin(2x)\,\sigma_z)}$$

Entropies: $S_I = 0$ (pure state)

$$S_{IA} = S_{IB} = \underline{-\tfrac{1}{2}(1+\sin(2x))\log\left(\tfrac{1}{2}(1+\sin(2x))\right) - \tfrac{1}{2}(1-\sin(2x))\log\left(\tfrac{1}{2}(1-\sin(2x))\right)}$$

$x = 0, \frac{\pi}{2}$ $\quad S_{IA} = S_{IB} = \log 2$; maximally entangled states

$x = \frac{\pi}{4}$ $\quad S_{IA} = S_{IB} = 0$, non-entangled, product state $|\psi\rangle = |+\rangle \otimes |-\rangle$

b) Case II

$$\hat{\rho}_{II} = \cos^2 x\, |1\rangle\langle 1| + \sin^2 x\, |2\rangle\langle 2|$$

$$\Rightarrow S_{II} = \underline{-\cos^2 x \log(\cos^2 x) - \sin^2 x \log(\sin^2 x)}$$

$\hat{\rho}_{II}$ obtained from $\hat{\rho}_{I}$ by deleting terms proportional to $\cos x \sin x = \frac{1}{2}\sin(2x)$:

$$\hat{\rho}_{II} = \frac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|) + \frac{1}{2}\cos(2x)(|+-\rangle\langle -+| + |-+\rangle\langle +-|)$$

$$\Rightarrow \underline{\hat{\rho}_{IIA} = \hat{\rho}_{IIB} = \frac{1}{2}\mathbb{1}} \Rightarrow \underline{S_{IIA} = S_{IIB} = \log 2}$$

$x = 0, \pi/2$ Same as in case I

$x = \pi/4$, $S_{II} = \log 2$; maximally mixed

$$\hat{\rho}_{II} = \frac{1}{2}(|+-\rangle\langle +-| + |-+\rangle\langle -+|)$$

separable (sum of product states) $\Rightarrow$ non-entangled

c) $\Delta_I = -S_{IA} = -S_{IB}$

is negative, unless $S_{IA} = S_{IB} = 0$,
which happens for $x = \pi/4$.

$$\Delta_{II} = S_{II} - \log 2$$

$S_{II} \leq \log 2$ since the Hilbert space is two-dimensional

$\Rightarrow \Delta_{II} \leq 0$, $\Delta_{II} = 0$ only when $S_{II} = \log 2$,

this happens only when $\underline{x = \pi/4}$ $\Rightarrow \cos^2 x = \sin^2 x = \frac{1}{2}$

Problem 2

a) Matrix elements of $\hat{x}$

$$x_{mn} = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle m|\hat{a}^{+}|n\rangle + \langle m|\hat{a}|n\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\delta_{m,n+1} + \sqrt{n}\,\delta_{m,n-1}\right)$$

Non-vanishing: $x_{n-1,n} = x_{n,n-1} = \sqrt{\frac{\hbar n}{2m\omega}}$

Photon emission: $|n\rangle \rightarrow |n-1\rangle \quad (E_n \rightarrow E_{n-1} + \hbar\omega)$

$$\Rightarrow \quad \underline{W_{n-1,n} = \frac{2\alpha\hbar}{3mc^2}\omega^2 n = \gamma n}$$

b) $$\frac{dp_n}{dt} = \langle n|\left(-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}] - \frac{1}{2}\gamma\left(\hat{a}^{+}\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^{+}\hat{a} - 2\hat{a}\hat{\rho}\hat{a}^{+}\right)\right)|n\rangle$$

$$\underline{= -\gamma\left(np_n - (n+1)p_{n+1}\right)}$$

$W_{n-1,n}$ = transition rate when state $|n\rangle$ occupied

$\Rightarrow \quad p_n = 1, \quad p_m = 0 \quad m \neq n$

With this assumption, conservation of probability gives $\frac{dp_n}{dt} = -W_{n-1,n}$

$= -\gamma n$ (from eq. (9))

consistent with eq. (8).

c) Excitation energy

$$E = Tr(\hat{H}_o \hat{\rho}) - \tfrac{1}{2}\hbar\omega$$

$$= \sum_n \hbar\omega(n+\tfrac{1}{2})\langle n|\hat{\rho}|n\rangle - \tfrac{1}{2}\hbar\omega$$

$$= \sum_n \hbar\omega\, n\, p_n$$

$$\Rightarrow \frac{dE}{dt} = \hbar\omega \sum_n n \frac{dp_n}{dt}$$

$$= -\gamma\hbar\omega \sum_n (n^2 p_n - n(n+1)p_{n+1})$$

$$= -\gamma\hbar\omega \sum_n (n^2 - n(n-1))\, p_n$$

$$= -\gamma\hbar\omega \sum_n n\, p_n$$

$$= \underline{-\gamma E}$$

Integrated

$$\frac{dE}{E} = -\gamma dt \Rightarrow \ln E = -\gamma t + \text{const}$$

$$\Rightarrow \underline{E(t) = E(0)e^{-\gamma t}}$$ exponential decay

Problem 3

a) $\mathrm{Tr}\,\hat{\rho} = 1 \Rightarrow N(\beta)^{-1} = \mathrm{Tr}(e^{-\beta H})$

$$= \sum_n e^{-\beta E_n}$$

$$E(\beta) = \mathrm{Tr}(\hat{H}\hat{\rho}) = N\,\mathrm{Tr}(\hat{H} e^{-\beta \hat{H}})$$

$$= -N \frac{\partial}{\partial\beta} \mathrm{Tr}(e^{-\beta\hat{H}}) = -N \frac{\partial}{\partial\beta} N^{-1}$$

$$= \frac{1}{N}\frac{\partial}{\partial\beta} \ln N = \underline{\frac{\partial}{\partial\beta} \ln N(\beta)}$$

Entropy: $S(\beta) = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho})$

$$= -\mathrm{Tr}(N e^{-\beta\hat{H}}(\ln N - \beta\hat{H}))$$

$$= -\ln N\, \mathrm{Tr}\hat{\rho} + \beta\, \mathrm{Tr}(\hat{H}\hat{\rho})$$

$$= -\ln N + \beta E(\beta)$$

$$= \underline{\beta \frac{\partial}{\partial\beta} \ln N(\beta) - \ln N(\beta)}$$

b) $\hat{H} = \frac{1}{2}\varepsilon\sigma_z \Rightarrow E_\pm = \pm\frac{1}{2}\varepsilon$

$$\Rightarrow N^{-1} = e^{\frac{1}{2}\varepsilon\beta} + e^{-\frac{1}{2}\varepsilon\beta} = 2\cosh(\tfrac{1}{2}\varepsilon\beta)$$

$$N(\beta) = \underline{\frac{1}{2\cosh(\frac{1}{2}\varepsilon\beta)}}$$

$$E(\beta) = -2\cosh(\tfrac{1}{2}\varepsilon\beta)\frac{1}{2\cosh^2(\frac{1}{2}\varepsilon\beta)} \sinh(\tfrac{1}{2}\varepsilon\beta)\cdot\tfrac{1}{2}\varepsilon$$

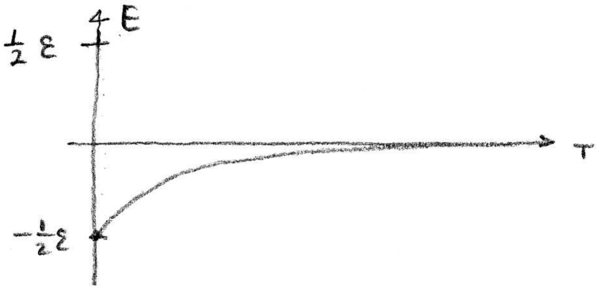
$$= \underline{-\tfrac{1}{2}\varepsilon\tanh(\tfrac{1}{2}\varepsilon\beta)}$$

$$\underline{S(\beta) = -\tfrac{1}{2}\varepsilon\beta\tanh(\tfrac{1}{2}\varepsilon\beta) + \ln(2\cosh(\tfrac{1}{2}\varepsilon\beta))}$$

$$E(\beta) = -\tfrac{1}{2}\varepsilon \tanh(\tfrac{1}{2}\varepsilon\beta)$$
$$= -\frac{1}{2}\varepsilon \, \frac{e^{\frac{1}{2}\varepsilon\beta} - e^{-\frac{1}{2}\varepsilon\beta}}{e^{\frac{1}{2}\varepsilon\beta} + e^{-\frac{1}{2}\varepsilon\beta}}$$

$$T \to 0 \Rightarrow \beta \to \infty \Rightarrow E(\beta) \simeq -\frac{1}{2}\varepsilon(1 - e^{-\varepsilon\beta}) \to -\frac{1}{2}\varepsilon$$

$$T \to \infty \Rightarrow \beta \to 0 \Rightarrow E(\beta) \simeq -\frac{1}{4}\varepsilon^2\beta = -\frac{1}{4}\frac{\varepsilon^2}{k_\beta T} \to 0$$



c) $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) \Rightarrow \vec{r} = Tr(\vec{\sigma}\hat{\rho})$

since $Tr\,\sigma_i = 0$ and $Tr(\sigma_i\sigma_j) = 2\delta_{ij}$

$$\vec{r} = N\,Tr(\vec{\sigma}\, e^{-\frac{1}{2}\varepsilon\beta\sigma_z})$$
$$= N\,Tr(\sigma_z e^{-\frac{1}{2}\varepsilon\beta\sigma_z})\,\vec{k}$$
$$= -\frac{2}{\varepsilon} N \frac{\partial}{\partial\beta}(Tr\, e^{-\frac{1}{2}\varepsilon\beta\sigma_z})\,\vec{k}$$
$$= -\frac{2}{\varepsilon} N \frac{\partial}{\partial\beta} N^{-1}\vec{k}$$
$$= -\frac{2}{\varepsilon} E(\beta)\,\vec{k}$$
$$= \underline{\tanh(\tfrac{1}{2}\varepsilon\beta)\vec{k}}$$

$\vec{r} = r\vec{k}$ with $r = -\frac{2}{\varepsilon}E(\beta)$

$T = 0\ (\beta = \infty)$: $r = 1$ pure state

$T \to \infty\ (\beta \to 0)$: $r \to 0$ maximally mixed

EXAM in FYS 4110/9110 Modern Quantum Mechanics 2015
Solutions

PROBLEM 1

a) Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega(\,\sigma_z \otimes \mathbb{1} - \,\mathbb{1} \otimes \sigma_z) + \hbar\lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{1}$$

Action on the basis states

$$\begin{aligned} \hat{H}|++\rangle &= \hat{H}|--\rangle = 0 \\ \hat{H}|+-\rangle &= \hbar\omega|+-\rangle + \hbar\lambda|-+\rangle \\ \hat{H}|-+\rangle &= -\hbar\omega|-+\rangle + \hbar\lambda|+-\rangle \end{aligned} \tag{2}$$

Matrix form of $\hat{H}$

$$H = \hbar\begin{pmatrix} \omega & \lambda \\ \lambda & -\omega \end{pmatrix} = \hbar a \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{3}$$

b) Eigenvalue equation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \epsilon \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{4}$$

Secular equation

$$\epsilon^2 - \cos^2\theta - \sin^2\theta = 0 \quad \Rightarrow \quad \epsilon = \pm 1 \equiv \epsilon_\pm \tag{5}$$

Energy eigenvalues

$$E_\pm = \pm\hbar a = \pm\hbar\sqrt{\omega^2 + \lambda^2} \tag{6}$$

Eigenvectors

$$\begin{aligned} \cos\theta\alpha_\pm + \sin\theta\beta_\pm &= \pm\alpha_\pm \\ \Rightarrow \quad \alpha_+/\beta_+ = (1+\cos\theta)/\sin\theta &= \cot\frac{\theta}{2} \\ \alpha_-/\beta_- = (-1+\cos\theta)/\sin\theta &= -\tan\frac{\theta}{2} \end{aligned} \tag{7}$$

$$\begin{aligned} \Rightarrow \quad |\psi_+\rangle &= \cos\frac{\theta}{2}|+-\rangle + \sin\frac{\theta}{2}|-+\rangle \\ |\psi_-\rangle &= \sin\frac{\theta}{2}|+-\rangle - \cos\frac{\theta}{2}|-+\rangle \end{aligned} \tag{8}$$

The states $|++\rangle$ and $|--\rangle$ are energy eigenstates with eigenvalues $E = 0$.

c) Product states

$$\hat{\rho}_1 = |++\rangle\langle++|\,, \quad \hat{\rho}_2 = |--\rangle\langle--| \tag{9}$$

have no entanglement. Reduced density operators

$$\hat{\rho}_1^A = \hat{\rho}_1^B = |+\rangle\langle+|\,, \quad \hat{\rho}_2^A = \hat{\rho}_2^B = |-\rangle\langle-| \tag{10}$$

Non-product states

$$\begin{aligned} \hat{\rho}_\pm = |\psi_\pm\rangle\langle\psi_\pm| &= \cos^2\frac{\theta}{2}|+-\rangle\langle+-| + \sin^2\frac{\theta}{2}|-+\rangle\langle+-| \\ &\pm \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}(|+-\rangle\langle-+| + |-+\rangle\langle+-|) \end{aligned} \tag{11}$$

Reduced density operators

$$\begin{aligned} \hat{\rho}_+^A = \hat{\rho}_-^B &= \cos^2\frac{\theta}{2}|+\rangle\langle+| + \sin^2\frac{\theta}{2}|-\rangle\langle-| \\ \hat{\rho}_-^A = \hat{\rho}_+^B &= \sin^2\frac{\theta}{2}|+\rangle\langle+| + \cos^2\frac{\theta}{2}|-\rangle\langle-| \end{aligned} \tag{12}$$

Entanglement entropies

$$S_\pm(\theta) = \cos^2\frac{\theta}{2}\log(\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\log(\sin^2\frac{\theta}{2}) \tag{13}$$

Minimum entanglement for $\theta = 0$ ($\lambda/\omega = 0$), with $S_\pm(0) = 0$, maximum entanglement for $\theta = \pm\pi/2$ ($\omega/\lambda = 0$), with $S_\pm(0) = \log 2$. This is identical to the maximum possible entanglement entropy in the two-spin system.

PROBLEM 2

a) Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{14}$$

In the Heisenberg picture

$$\dot{\hat{a}}_H = \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right]_H = -i\omega_0\hat{a}_H - i\lambda e^{-i\omega t}\mathbb{1} \tag{15}$$

gives

$$\ddot{\hat{a}}_H = \frac{i}{\hbar}\left[\hat{H}, \dot{\hat{a}}_H\right] + \frac{\partial\dot{\hat{a}}_H}{\partial t} = -\omega_0^2\hat{a}_H - \lambda(\omega_0 + \omega)e^{-i\omega t}\mathbb{1} \tag{16}$$

which gives $C = -\lambda(\omega_0 + \omega)$.

b) Assume

$$\hat{a}_H = \hat{a}e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\mathbb{1} \tag{17}$$

Differentiation gives

$$\begin{aligned}\ddot{\hat{a}}_H &= -\omega_0^2 \hat{a} e^{-i\omega_0 t} - D(\omega^2 e^{-i\omega t} - \omega_0^2 e^{-i\omega_0 t}) \\ &= -\omega_0^2 \hat{a}_H - (\omega^2 - \omega_0^2) D e^{-i\omega t}\end{aligned} \tag{18}$$

which is of the form (16) with

$$D = \frac{\lambda}{\omega - \omega_0} \tag{19}$$

c) Time evolution

$$\begin{aligned}|\psi(0)\rangle &= |0\rangle\,, \quad \hat{a}|0\rangle = 0 \\ |\psi(t)\rangle &= \hat{\mathcal{U}}(t)|\psi(0)\rangle\end{aligned} \tag{20}$$

gives

$$\begin{aligned}\hat{a}|\psi(t)\rangle &= \hat{\mathcal{U}}(t)\hat{\mathcal{U}}^\dagger(t)\,\hat{a}\,\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\ \hat{a}_H(t)\,|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\ (\hat{a}e^{-i\omega_0 t} + D(e^{-i\omega t} - e^{-i\omega_0 t})\,|\psi(0)\rangle \\ &= \frac{\lambda}{\omega - \omega_0}\,(e^{-i\omega t} - e^{-i\omega_0 t})\,|\psi(t)\rangle\end{aligned} \tag{21}$$

This shows that $|\psi(t)\rangle$ is a coherent state with time dependent complex parameter $z(t)$, and with real part $x(t)$, given by

$$z(t) = \frac{\lambda}{\omega - \omega_0}(e^{-i\omega t} - e^{-i\omega_0 t})\,, \quad x(t) = \frac{\lambda}{\omega - \omega_0}(\cos\omega t - \cos\omega_0 t) \tag{22}$$

The time evolution of the coordinate $x(t)$ is the same as for the classical driven harmonic oscillator,

$$\ddot{x} + \omega_0^2 x = -\lambda(\omega_0 + \omega)\cos\omega t \tag{23}$$

PROBLEM 3

a) Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\lambda(\hat{a}^\dagger\sigma_- + \hat{a}\sigma_+) \tag{24}$$

Action on the states $|-,1\rangle$ and $|+,0\rangle$,

$$\begin{aligned}\hat{H}|-,1\rangle &= \frac{1}{2}\hbar\,(\omega|-,1\rangle + \lambda|+,0\rangle) \\ \hat{H}|+,0\rangle &= \frac{1}{2}\hbar\,(\omega|+,0\rangle + \lambda|-,1\rangle)\end{aligned} \tag{25}$$

Matrix form

$$\hat{H} = \frac{1}{2}\hbar\omega\mathbb{1} + \frac{1}{2}\hbar\lambda\sigma_x \tag{26}$$

Eigenvalues for σ_x are ± 1, gives energy eigenvalues

$$E_{\pm} = \frac{1}{2}\hbar(\omega \pm \lambda) \tag{27}$$

Energy eigenstates

$$|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|-,1\rangle \pm |+,0\rangle)\,, \quad \hat{H}|\psi_{\pm}\rangle = E_{\pm}|\psi_{\pm}\rangle \tag{28}$$

Time dependent state

$$|\psi(t)\rangle = c_+ e^{-\frac{i}{\hbar}E_+ t}|\psi_+\rangle + c_- e^{-\frac{i}{\hbar}E_- t}|\psi_-\rangle \tag{29}$$

Initial condition $|\psi(0)\rangle = |-,1\rangle$ implies $c_+ = c_- = \frac{1}{\sqrt{2}}$,

$$|\psi(t)\rangle = e^{-\frac{i}{2}t}(\cos(\frac{\lambda}{2}t)|-,1\rangle - i(\sin(\frac{\lambda}{2}t)|+,0\rangle) \tag{30}$$

which gives $\epsilon = -\omega/2$ and $\Omega = \lambda/2$.

b) The Lindblad equation gives for the occupation probability of the ground state

$$\frac{dp_g}{dt} = -\frac{i}{\hbar}\langle -,0|\left[\hat{H},\hat{\rho}\right]|-,0\rangle + \gamma\langle -,0|\hat{a}\hat{\rho}\hat{a}^{\dagger}|-,0\rangle = \gamma\langle -,1|\hat{\rho}|-,1\rangle \tag{31}$$

When a photon is present in the cavity, $\langle -,1|\hat{\rho}|-,1\rangle \neq 0$, this gives $\dot{p}_g > 0$, which implies that the occupation probability of the ground state increases until there is no photon in the cavity, $\langle -,1|\hat{\rho}|-,1\rangle = 0$.

c) Evaluation of the matrix elements of the Lindblad equation in the subspace spanned by $|-,1\rangle$ and $|+,0\rangle$ gives

$$\begin{aligned}
\dot{p}_1 &= -\frac{i}{2}\lambda(\langle +,0|\hat{\rho}|-,1\rangle - \langle -,1|\hat{\rho}|+,0\rangle) - \gamma p_1 \\
\dot{p}_0 &= -\frac{i}{2}\lambda(\langle -,1|\hat{\rho}|+,0\rangle - \langle +,0|\hat{\rho}|-,1\rangle) \\
\dot{b} &= -\frac{i}{2}\lambda(\langle +,0|\hat{\rho}|+,0\rangle - \langle -,1|\hat{\rho}|-,1\rangle) - \frac{1}{2}\gamma b
\end{aligned} \tag{32}$$

which simplifies to

$$\begin{aligned}
\dot{p}_1 &= -\gamma p_1 - \lambda b \\
\dot{p}_0 &= \lambda b \\
\dot{b} &= -\frac{1}{2}\gamma b + \frac{1}{2}\lambda(p_1 - p_0)
\end{aligned} \tag{33}$$

Expected time evolution: Exponentially damped oscillations between the states $|-,1\rangle$ and $|+,0\rangle$, with the system ending in the photon less ground state $|-,0\rangle$.

Exam FYS4110, fall semester 2016
Solutions

PROBLEM 1
a) Matrix elements of $\hat{H}$ in the two-dimensional subspace

$$\begin{aligned}\hat{H}|0,+1\rangle &= \frac{1}{2}\hbar(\omega_0+\omega_1)|0,+1\rangle + \lambda\hbar|1,-1\rangle \\ \hat{H}|1,-1\rangle &= \frac{1}{2}\hbar(3\omega_0-\omega_1)|0,+1\rangle + \lambda\hbar|0,+1\rangle\end{aligned} \tag{1}$$

In matrix form

$$H = \frac{1}{2}\hbar\begin{pmatrix}\omega_0+\omega_1 & 2\lambda \\ 2\lambda & 3\omega_0-\omega_1\end{pmatrix} = \frac{1}{2}\hbar\Delta\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix} + \epsilon\hbar\mathbb{1} \tag{2}$$

which gives

$$\Delta\cos\theta = \omega_1-\omega_0\,, \quad \Delta\sin\theta = 2\lambda\,, \quad \epsilon = \omega_0 \tag{3}$$

and from this

$$\Delta = \sqrt{(\omega_1-\omega_0)^2+4\lambda^2} \tag{4}$$

and

$$\cos\theta = \frac{\omega_1-\omega_0}{\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}}\,, \quad \sin\theta = \frac{2\lambda}{\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}} \tag{5}$$

b) Eigenvalue problem for the matrix

$$\begin{aligned}\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} &= \delta\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ \begin{vmatrix}\cos\theta-\delta & \sin\theta \\ \sin\theta & -\cos\theta-\delta\end{vmatrix} &= 0 \\ \Rightarrow \quad \delta^2-\cos^2\theta-\sin^2\theta &= 0 \quad \Rightarrow \quad \delta = \pm 1\end{aligned} \tag{6}$$

Energy eigenvalues

$$E_\pm = \hbar(\epsilon \pm \frac{1}{2}\Delta) = \hbar\left(\omega_0 \pm \frac{1}{2}\sqrt{(\omega_1-\omega_0)^2+4\lambda^2}\right) \tag{7}$$

Eigenvectors

$$(\cos\theta \mp 1)\,\alpha + \sin\theta\,\beta = 0 \quad \Rightarrow \quad \frac{\beta}{\alpha} = \pm\frac{1\mp\cos\theta}{\sin\theta} \tag{8}$$

This gives

$$\begin{pmatrix}\alpha_\pm \\ \beta_\pm\end{pmatrix} = N_\pm\begin{pmatrix}\pm\sin\theta \\ 1\mp\cos\theta\end{pmatrix} \tag{9}$$

with normalization factor

$$N_\pm^2 = \sin^2\theta + (1\mp\cos\theta)^2 = 2(1\mp\cos\theta) \tag{10}$$

Finally

$$\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix} = \frac{1}{\sqrt{2(1 \mp \cos\theta)}} \begin{pmatrix} \pm\sin\theta \\ 1 \mp \cos\theta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm\sqrt{1 \pm \cos\theta} \\ \sqrt{1 \mp \cos\theta} \end{pmatrix} \tag{11}$$

and in bra-ket form

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} \left(\pm\sqrt{1 \pm \cos\theta}\, |0, +1\rangle + \sqrt{1 \mp \cos\theta}\, |1, -1\rangle \right) \tag{12}$$

c) Density operator

$$\begin{aligned} \hat{\rho}_\pm &= \frac{1}{2}(1 \pm \cos\theta)(|0\rangle\langle 0| \otimes |+1\rangle\langle +1|) + \frac{1}{2}(1 \mp \cos\theta)(|1\rangle\langle 1| \otimes |-1\rangle\langle -1|) \\ &\pm \frac{1}{2}\sin\theta(|0\rangle\langle 1| \otimes |+1\rangle\langle -1| + |1\rangle\langle 0| \otimes |-1\rangle\langle +1|) \end{aligned} \tag{13}$$

Reduced density operators

$$\begin{aligned} \text{position}: \quad & \hat{\rho}^p_\pm = \mathrm{Tr}_s\, \hat{\rho}_\pm = \frac{1}{2}(1 \pm \cos\theta)|0\rangle\langle 0| + \frac{1}{2}(1 \mp \cos\theta)|1\rangle\langle 1| \\ \text{spin}: \quad & \hat{\rho}^s_\pm = \mathrm{Tr}_p\, \hat{\rho}_\pm = \frac{1}{2}(1 \pm \cos\theta)|+1\rangle\langle +1| + \frac{1}{2}(1 \mp \cos\theta)|-1\rangle\langle -1| \end{aligned} \tag{14}$$

Entanglement entropy

$$\begin{aligned} S^p_\pm = S^s_\pm &= -[\frac{1}{2}(1 - \cos\theta)\log(\frac{1}{2}(1 - \cos\theta)) + \frac{1}{2}(1 + \cos\theta)\log(\frac{1}{2}(1 + \cos\theta)] \\ &= -[\cos^2\frac{\theta}{2}\log(\cos^2\frac{\theta}{2}) + \sin^2\frac{\theta}{2}\log(\sin^2\frac{\theta}{2})] \equiv S \end{aligned} \tag{15}$$

Maximum entanglement

$$\theta = \frac{\pi}{2}: \quad \cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2} = \frac{1}{2} \quad \Rightarrow \quad S = \log 2 \tag{16}$$

Minimum entanglement

$$\begin{aligned} \theta = 0: &\quad \cos^2\frac{\theta}{2} = 1,\ \sin^2\frac{\theta}{2} = 0 \quad \Rightarrow \quad S = 0 \\ \theta = \pi: &\quad \cos^2\frac{\theta}{2} = 0,\ \sin^2\frac{\theta}{2} = 1 \quad \Rightarrow \quad S = 0 \end{aligned} \tag{17}$$

PROBLEM 2

a) Change of variables

$$\begin{aligned} \hat{c}^\dagger\hat{c} &= \mu^2\hat{a}^\dagger\hat{a} + \nu^2\hat{b}^\dagger\hat{b} + \mu\nu(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \\ \hat{d}^\dagger\hat{d} &= \nu^2\hat{a}^\dagger\hat{a} + \mu^2\hat{b}^\dagger\hat{b} - \mu\nu(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \\ \Rightarrow \quad \omega_c\hat{c}^\dagger\hat{c} + \omega_d\hat{d}^\dagger\hat{d} &= (\mu^2\omega_c + \nu^2\omega_d)\hat{a}^\dagger\hat{a} + (\nu^2\omega_c + \mu^2\omega_d)\hat{b}^\dagger\hat{b} \\ &+ \mu\nu(\omega_c - \omega_d)(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) \end{aligned} \tag{18}$$

To get the correct form for the Hamiltonian, define ω_c, ω_d, μ and ν so that the following equations are satisfied

$$\begin{array}{lrcl} \mathrm{I} & \mu^2+\nu^2 & = & 1 \\ \mathrm{II} & \mu^2\omega_c+\nu^2\omega_d & = & \omega \\ \mathrm{III} & \nu^2\omega_c+\mu^2\omega_d & = & \omega \\ \mathrm{IV} & \mu\nu(\omega_c-\omega_d) & = & \lambda \end{array} \tag{19}$$

From I, II and III follows

$$\begin{array}{lrcl} \mathrm{IIb} & \frac{1}{2}(\omega_c+\omega_d) & = & \omega \\ \mathrm{IIIb} & (\mu^2-\nu^2)(\omega_c-\omega_d) & = & 0 \end{array} \tag{20}$$

Since $\omega_c \neq \omega_d$ from IV, we have $\mu^2=\nu^2=1/2$, and therefore (by convenient choice of sign factors) $\mu=\nu=1/\sqrt{2}$. Inserted in IV this gives

$$\mathrm{IVb} \qquad \frac{1}{2}(\omega_c-\omega_d) \;=\; \lambda \tag{21}$$

which together with IIb gives

$$\omega_c=\omega+\lambda\,,\quad \omega_d=\omega-\lambda \tag{22}$$

Commutation relations

$$\begin{aligned} \left[\hat{c},\hat{c}^\dagger\right] &= \mu^2\left[\hat{a},\hat{a}^\dagger\right]+\nu^2\left[\hat{b},\hat{b}^\dagger\right]=(\mu^2+\nu^2)\mathbb{1}=\mathbb{1} \\ \left[\hat{c},\hat{d}^\dagger\right] &= -\mu\nu(\left[\hat{a},\hat{a}^\dagger\right]-\left[\hat{b},\hat{b}^\dagger\right])=0 \end{aligned} \tag{23}$$

Similar evaluations of other commutators show that the two sets of ladder operators satify the standard commutation rules for two independent harmonic oscillators.

b) Time evolution of a coherent state

$$\begin{aligned} |\psi(t)\rangle &= \hat{\mathcal{U}}(t)|\psi(0)\rangle\,,\quad \hat{\mathcal{U}}(t)=\exp[-i(\omega_c\hat{c}^\dagger\hat{c}+\omega_d\hat{d}^\dagger\hat{d}+\omega\mathbb{1})] \\ \Rightarrow\quad \hat{c}|\psi(t)\rangle &= \hat{\mathcal{U}}(t)\hat{\mathcal{U}}(t)^{-1}\hat{c}\hat{\mathcal{U}}(t)|\psi(0)\rangle \\ &= \hat{\mathcal{U}}(t)\,e^{i\omega_c t\hat{c}^\dagger\hat{c}}\,\hat{c}\,e^{-i\omega_c t\hat{c}^\dagger\hat{c}}\,|\psi(0)\rangle \\ &= e^{-i\omega_c t}\hat{\mathcal{U}}(t)\,\hat{c}\,|\psi(0)\rangle \\ &= e^{-i\omega_c t}z_{c0}\,|\psi(0)\rangle \end{aligned} \tag{24}$$

$|\psi(t)\rangle$ is thus a coherent state of the c-oscillator with eigenvalue $z_c(t)=e^{-i\omega_c t}z_{c0}$. Simlar result is valid for the d- oscillator with $z_d(t)=e^{-i\omega_d t}z_{d0}$.

c) Since all the operators $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ commute, they have a common set of eigenvalues. This implies that a state which is a coherent state of $\hat{c}$, and $\hat{d}$ will also be a coherent state of $\hat{a}$ and $\hat{b}$. As follows from a) we have

$$\hat{a}=\frac{1}{\sqrt{2}}(\hat{c}-\hat{d})\,,\quad \hat{b}=\frac{1}{\sqrt{2}}(\hat{c}+\hat{d}) \tag{25}$$

The corresponding relations between the eigenvalues are

$$\begin{aligned} z_a(t) &= \frac{1}{\sqrt{2}}(z_c(t) - z_d(t)) \\ &= \frac{1}{\sqrt{2}}(e^{-i\omega_c t} z_{c0} - e^{-i\omega_d t} z_{d0}) \\ &= \frac{1}{2} e^{-i\omega t}(e^{-i\lambda t}(z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0})) \\ &= \frac{1}{2} e^{-i\omega t}(\cos(\lambda t) z_{a0} - i \sin(\lambda t) z_{b0}) \end{aligned} \tag{26}$$

and similarly

$$\begin{aligned} z_b(t) &= \frac{1}{2} e^{-i\omega t}(-e^{-i\lambda t}(z_{a0} + z_{b0}) + e^{i\lambda t}(z_{a0} - z_{b0})) \\ &= \frac{1}{2} e^{-i\omega t}(i \sin(\lambda t) z_{a0} + \cos(\lambda t) z_{b0}) \end{aligned} \tag{27}$$

PROBLEM 3

a) Time derivatives of matrix elements

$$\begin{aligned} \text{I} \qquad \dot{p}_e &= \langle e|\frac{d\hat{\rho}}{dt}|e\rangle = -\gamma p_e + \gamma' p_g \\ \text{II} \qquad \dot{p}_g &= \langle g|\frac{d\hat{\rho}}{dt}|g\rangle = -\gamma' p_g + \gamma p_e \\ \text{III} \qquad \dot{b} &= \langle e|\frac{d\hat{\rho}}{dt}|g\rangle = [\frac{i}{\hbar}\Delta E - \frac{1}{2}(\gamma + \gamma')]b \end{aligned} \tag{28}$$

From I and II follows $\frac{d}{dt}(p_e + p_g = 0)$, the sum of occupation probabilities is constant.

b) Conditions satisfied by the density operator

$$\begin{aligned} 1) \qquad \hat{\rho} &= \hat{\rho}^\dagger \\ 2) \qquad \hat{\rho} &\geq 0 \\ 3) \qquad \mathrm{Tr}\,\hat{\rho} &= 1 \end{aligned} \tag{29}$$

1) implies that p_e and p_g are real, which is consistent with the interpretation of these as probabilities. 3) gives the normalization $p_e + p_g = 1$. 2) means that the eigenvalues of $\hat{\rho}$ are non-negative. To see the implication of this we find the eigenvalues from the secular equation

$$\begin{aligned} &\begin{vmatrix} p_e - \lambda & b \\ b^* & p_g - \lambda \end{vmatrix} = 0 \\ \Rightarrow \qquad & \lambda^2 - \lambda + p_e p_g - |b|^2 = 0 \\ \Rightarrow \qquad & \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 + 4(|b|^2 - p_e p_g)} \end{aligned} \tag{30}$$

Positivity of λ_- then requires $|b|^2 \leq p_e p_g$.

c) At thermal equilibrium we have $\dot{p}_e = \dot{p}_g = \dot{b} = 0$. I then implies

$$\gamma p_e = \gamma' p_g \quad \Rightarrow \quad \frac{p_e}{p_g} = \frac{\gamma'}{\gamma} = e^{-\Delta E/kT} \tag{31}$$

Using $p_g = 1 - p_e$ we find

$$
\begin{aligned}
p_e = \frac{\gamma'/\gamma}{1+\gamma'/\gamma} &= \frac{1}{1+e^{\Delta E/kT}} \\
p_g = \frac{1}{1+\gamma'/\gamma} &= \frac{1}{1+e^{-\Delta E/kT}}
\end{aligned}
\tag{32}
$$

From III follows $\dot{b} = 0 \Rightarrow b = 0$.

d) From the initial values $p_e(0) = 1$, $p_g(0) = 0$, and the constraint on $|b|^2$ follows

$$
|b(0)|^2 \leq p_e(0)p_g(0) = 0 \quad \Rightarrow \quad b(0) = 0 \tag{33}
$$

We apply in the following the general formula

$$
\dot{x} = ax \quad \Rightarrow \quad x(t) = e^{at}x(0) \tag{34}
$$

For b this means

$$
b(t) = e^{-\frac{i}{b}\Delta E - \frac{1}{2}(\gamma+\gamma')t}\, b(0) = 0 \tag{35}
$$

With $p_e = 1 - p_g$ eq. II gives for p_g

$$
\dot{p}_g = -(\gamma+\gamma')p_g + \gamma = -(\gamma+\gamma')(p_g - \frac{1}{1+\gamma'/\gamma}) \tag{36}
$$

or

$$
\frac{d}{dt}(p_g - \frac{1}{1+\gamma'/\gamma}) = -(\gamma+\gamma')p_g + \gamma = -(\gamma+\gamma')(p_g - \frac{1}{1+\gamma'/\gamma}) \tag{37}
$$

Integrating the equation gives

$$
p_g(t) - \frac{1}{1+\gamma'/\gamma} = e^{-(\gamma+\gamma')t}(p_g(0) - \frac{1}{1+\gamma'/\gamma}) \tag{38}
$$

which with $p_g(0) = 1$ is solved to

$$
p_g(t) = \frac{1}{1+\gamma'/\gamma}(1 + (\gamma'/\gamma)e^{-(\gamma+\gamma')t}) \tag{39}
$$

and for $p_e = 1 - p_g$ gives

$$
p_e(t) = \frac{\gamma'/\gamma}{1+\gamma'/\gamma}(1 + e^{-(\gamma+\gamma')t}) \tag{40}
$$

We note that the above expressions reproduce correctly, in the limit $t \to \infty$, the values for p_e and p_g at thermal equilibrium.

The limit $T \to 0$ gives $\gamma'/\gamma \to 0$. This gives $p_g(t) \to 1$ and $p_e(t) \to 0$ consistent with the fact that the system remains in the ground state when $T = 0$. In the limit $T \to \infty$ we have $\gamma'/\gamma \to 1$, which gives

$$
\begin{aligned}
p_g(t) &\to \frac{1}{2}(1 + e^{-2\gamma t}) \\
p_e(t) &\to \frac{1}{2}(1 - e^{-2\gamma t})
\end{aligned}
\tag{41}
$$

In this case the time evolution gives $\lim_{t\to\infty} p_e = \lim_{t\to\infty} p_g = \frac{1}{2}$.

Fys 4110 exam 2017 Solutions.

Problem 1.

a) $$H = \frac{\hbar}{2} g\, \sigma_z^A \otimes \sigma_z^B = \frac{\hbar}{2} g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = e^{-\frac{i}{\hbar} H t} = \begin{pmatrix} z^* & 0 & 0 & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z^* \end{pmatrix} \quad \text{where } z = e^{\frac{igt}{2}}$$

$|z| = 1$.

b) Alternative 1 (Brute force)

$$|\psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

$$|\psi(t)\rangle = U|\psi(0)\rangle = \begin{pmatrix} z^* ac \\ z\, ad \\ z\, bc \\ z^* bd \end{pmatrix}$$

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} z^* ac \\ z\, ad \\ z\, bc \\ z^* bd \end{pmatrix} \left(z a^* c^*,\ z^* a^* d^*,\ z^* b^* c^*,\ z b^* d^* \right)$$

$$= \begin{pmatrix} |ac|^2 & z^{*2}|a|^2 c d^* & z^{*2} a b^* |c|^2 & a b^* c d^* \\ z^2 |a|^2 c^* d & |ad|^2 & a b^* c^* d & z^2 a b^* |d|^2 \\ z^2 a^* b |c|^2 & a^* b c d^* & |bc|^2 & z^2 |b|^2 c d^* \\ a^* b c^* d & z^{*2} a^* b |d|^2 & z^{*2} |b|^2 c^* d & |bd|^2 \end{pmatrix}$$

$$\rho_A = \mathrm{Tr}_B\, \rho = \begin{pmatrix} |a|^2 & a b^* \left(z^{*2}|c|^2 + z^2 |d|^2 \right) \\ a^* b \left(z^2 |c|^2 + z^{*2} |d|^2 \right) & |b|^2 \end{pmatrix}$$

$$\rho_B = \mathrm{Tr}_A\, \rho = \begin{pmatrix} |c|^2 & c d^* \left(z^{*2}|a|^2 + z^2 |b|^2 \right) \\ c^* d \left(z^2 |a|^2 + z^{*2} |b|^2 \right) & |d|^2 \end{pmatrix}$$

Alternative 2 (More sophisticated, but not really simpler...)

With $z = x + iy$ we find

$$U = x\, \mathbb{1}^A \otimes \mathbb{1}^B - iy\, \sigma_z^A \otimes \sigma_z^B$$

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| = U \underbrace{|\psi(0)\rangle\langle\psi(0)|}_{\rho(0) = \rho^A(0) \otimes \rho^B(0)} U^+$$

Let $\rho^A(0) = \frac{1}{2}(\mathbb{1} + \vec{m}\cdot\vec{\sigma})$ $\qquad \rho^B(0) = \frac{1}{2}(\mathbb{1} + \vec{n}\cdot\vec{\sigma})$

$$\rho(t) = (x\, \mathbb{1}^A \otimes \mathbb{1}^B - iy\, \sigma_z^A \otimes \sigma_z^B)\rho^A(0) \otimes \rho^B(0)(x\, \mathbb{1}^A \otimes \mathbb{1}^B + iy\, \sigma_z^A \otimes \sigma_z^B)$$

$$= x^2 \rho^A(0) \otimes \rho^B(0) + y^2 \sigma_z^A \otimes \sigma_z^B\, \rho^A(0) \otimes \rho^B(0)\, \sigma_z^A \otimes \sigma_z^B$$
$$+ ixy\left[\rho^A(0) \otimes \rho^B(0)\, \sigma_z^A \otimes \sigma_z^B - \sigma_z^A \otimes \sigma_z^B\, \rho^A(0) \otimes \rho^B(0)\right]$$

$$= x^2 \rho^A(0) \otimes \rho^B(0) + y^2\, \sigma_z^A \rho^A(0) \sigma_z^A \otimes \sigma_z^B \rho^B(0) \sigma_z^B$$
$$+ ixy\left[\rho^A(0)\sigma_z^A \otimes \rho^B(0)\sigma_z^B - \sigma_z^A \rho^A(0) \otimes \sigma_z^B \rho^B(0)\right]$$

We have

$$\mathrm{Tr}\, \rho^A(0) = 1$$

$$\mathrm{Tr}\, \sigma_z^A \rho^A(0) \sigma_z^A = \frac{1}{2}\mathrm{Tr}\, \sigma_z^A (\mathbb{1} + \vec{m}\cdot\vec{\sigma})\sigma_z^A = 1$$

$$\mathrm{Tr}\, \rho^A(0)\sigma_z^A = \frac{1}{2}\mathrm{Tr}(\sigma_z^A + \vec{m}\cdot\vec{\sigma}\, \sigma_z^A) = m_z = \mathrm{Tr}\, \sigma_z^A \rho^A(0)$$

and similar for system B

$$\Rightarrow \rho^A(t) = \mathrm{Tr}_B\, \rho = x^2 \rho^A(0) + y^2 \sigma_z^A \rho^A(0) \sigma_z^A + ixy [\rho_A(0), \sigma_z^A]$$

$$= \frac{1}{2}\Big[\mathbb{1} + (m_x \cos gt - m_y n_z \sin gt)\, \sigma_x^A$$

$$+ (m_y \cos gt + m_x n_z \sin gt)\, \sigma_y^A + m_z\, \sigma_z^A \Big]$$

$$\rho^B(t) = \frac{1}{2}\Big[\mathbb{1} + (n_x \cos gt - n_y m_z \sin gt)\, \sigma_x^B$$

$$+ (n_y \cos gt + \cancel{m_x m_z} \sin gt)\, \sigma_y^B + n_z\, \sigma_z^A \Big]$$

c) Alternative 1

Using $z^2 = e^{igt} = \cos gt + i \sin gt$ and $a = b = \frac{1}{\sqrt{2}}$:

$$\rho^A = \frac{1}{2} \begin{pmatrix} 1 & \cos gt\, (\underbrace{|c|^2 + |d|^2}_{1}) - i \sin gt\, (\underbrace{|c|^2 - |d|^2}_{n_z}) \\ c.c. & 1 \end{pmatrix}$$

$$= \frac{1}{2}\left(\mathbb{1} + \cos gt\, \sigma_x^A + n_z \sin gt\, \sigma_y^A \right)$$

$$\Rightarrow \quad m_x(t) = \cos gt \qquad m_y(t) = n_z \sin gt \qquad m_z(t) = 0$$

$$m_x(t)^2 + \left(\frac{m_y(t)}{n_z} \right)^2 = 1 \qquad \Rightarrow \text{ellipse}$$

Alternative 2.

$$\rho^A(0) = \begin{pmatrix} a \\ b \end{pmatrix} (a^* \; b^*) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1\,1) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\left(\mathbb{1} + \sigma_x \right)$$

$$\Rightarrow m_x = 1, \quad m_y = m_z = 0$$

$$\rho^A(t) = \frac{1}{2}\left(\mathbb{1} + \cos gt\, \sigma_x^A + n_z \sin gt\, \sigma_y^A \right)$$

d) Maximal entanglement when the Bloch-vector is shortest $\Rightarrow$ $gt = \frac{\pi}{2}$ $\cos gt = 0$ $\sin gt = 1$.

$$\rho^A(t) = \frac{1}{2}(\mathbb{1} + n_z \sigma_y^A) = \frac{1}{2}\begin{pmatrix} 1 & -in_z \\ in_z & 1 \end{pmatrix}$$

Eigenvalues: $(\frac{1}{2} - \lambda)^2 - (\frac{n_z}{2})^2 = 0$ $\rightarrow \lambda_\pm = \frac{1}{2}(1 \pm n_z)$

$$S_{max} = -\frac{1+n_z}{2}\ln\frac{1+n_z}{2} - \frac{1-n_z}{2}\ln\frac{1-n_z}{2}$$

$$= \ln 2 - \frac{1}{2}\left[(1+n_z)\ln(1+n_z) + (1-n_z)\ln(1-n_z)\right] = \begin{cases} 0 & n_z = \pm 1 \\ \ln 2 & n_z = 0 \end{cases}$$

Problem 2

a) $S(\zeta) = e^{-\frac{1}{2}(\zeta a^2 - \zeta^* a^{+2})}$ $\qquad B = \frac{1}{2}(\zeta a^2 - \zeta^* a^{+2})$

$B^+ = -B$

$$S^+ a S = e^B a e^{-B} = a + [B, a] + \frac{1}{2}[B,[B,a]] + \dots$$

$$[B, a] = -\frac{1}{2}\zeta^*[a^{+2}, a] = -\frac{1}{2}\zeta^*(a^+[a^+, a] + [a^+, a]a^+) = \zeta^* a^+$$

$$[B, a^+] = \frac{1}{2}\zeta[a^2, a^+] = \frac{1}{2}\zeta(a[a, a^+] + [a, a^+]a) = \zeta a$$

$$S^+ a S = a + \zeta^* a^+ + \frac{1}{2}\zeta^*\zeta a + \frac{1}{3!}\zeta^{*2}\zeta a^+ + \frac{1}{4!}\zeta^{*2}\zeta^2 a + \dots$$

$$= \left[1 + \frac{1}{2!}|\zeta|^2 + \frac{1}{4!}|\zeta|^4 + \dots\right] a + \left[\zeta^* + \frac{1}{3!}\zeta^{*2}\zeta + \frac{1}{5!}\zeta^{*3}\zeta^2 + \dots\right] a^+$$

$$= \left[1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \dots\right] a + e^{-i\phi}\left[r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \dots\right] a^+$$

$$= \cosh r \, a + e^{-i\phi}\sinh r \, a^+$$

$$S^+ a^+ S = \cosh r \, a^+ + e^{i\phi}\sinh r \, a$$

b) $\langle sq_{rs}|x|sq_{rs}\rangle = \langle 0|S^{+}xS|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|S^{+}(a^{+}+a)S|0\rangle$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle 0|(\cosh r + e^{-i\phi}\sinh r)a^{+} + (\cosh r + e^{i\phi}\sinh r)a|0\rangle = 0$$

$\langle sq_{rs}|p|sq_{rs}\rangle = \langle 0|S^{+}pS|0\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle 0|S^{+}(a^{+}-a)S|0\rangle$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\langle 0|(\cosh r - e^{-i\phi}\sinh r)a^{+} - (\cosh r - e^{i\phi}\sinh r)a|0\rangle = 0$$

$\Delta x^2 = \langle sq_{rs}|x^2|sq_{rs}\rangle = \langle 0|S^{+}xSS^{+}xS|0\rangle$

$$= \frac{\hbar}{2m\omega}(\cosh r + e^{i\phi}\sinh r)(\cosh r + e^{-i\phi}\sinh r)$$

$$= \frac{\hbar}{2m\omega}\Big[\underbrace{\cosh^2 r + \sinh^2 r}_{\cosh 2r} + \underbrace{\cosh r\sinh r}_{\frac{1}{2}\sinh 2r}\underbrace{(e^{i\phi}+e^{-i\phi})}_{2\cos\phi}\Big]$$

$$= \frac{\hbar}{2m\omega}(\cosh 2r + \sinh 2r\cos\phi)$$

$\Delta p^2 = \langle sq_{rs}|p^2|sq_{rs}\rangle = \langle 0|S^{+}pSS^{+}pS|0\rangle$

$$= \frac{\hbar m\omega}{2}(\cosh r - e^{i\phi}\sinh r)(\cosh r - e^{-i\phi}\sinh r)$$

$$= \frac{\hbar m\omega}{2}\left[\cosh^2 r + \sinh^2 r - \cosh r\sinh r(e^{i\phi}+e^{-i\phi})\right]$$

$$= \frac{\hbar m\omega}{2}(\cosh 2r - \sinh 2r\cos\phi)$$

8) $\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\cosh^2 r - \sinh^2 r \cos^2\phi}$

$$= \frac{\hbar}{2}\sqrt{\cosh^2 r - \sinh^2 r\,(1-\sin^2\phi)}$$

$$= \frac{\hbar}{2}\sqrt{1 + \sinh^2 r \sin^2\phi}$$

Minimal uncertainty: $\Delta x \Delta p = \frac{\hbar}{2}$

$\rightarrow \sin\phi = 0 \quad \rightarrow \quad \phi = n\pi$

9) For $\phi = n\pi$:

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\cosh 2r + (-1)^n \sinh 2r} = \sqrt{\frac{\hbar}{2m\omega}}\, e^{(-1)^n r}$$

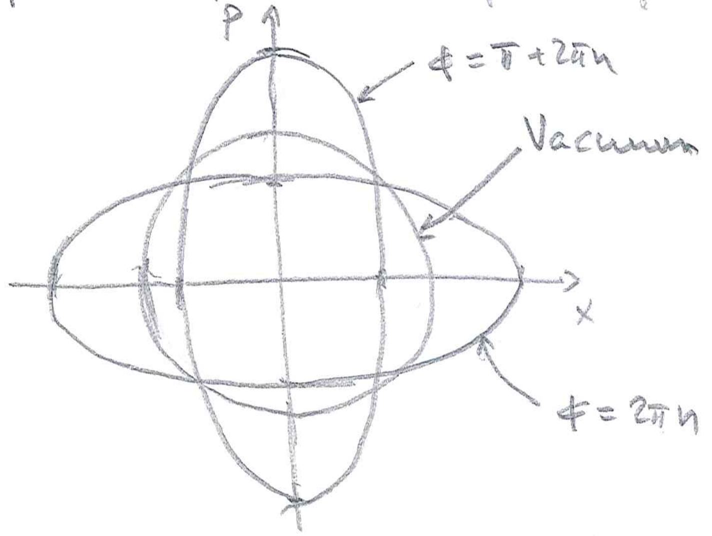
$$\Delta p = \sqrt{\frac{\hbar m\omega}{2}}\sqrt{\cosh 2r - (-1)^n \sinh 2r} = \sqrt{\frac{\hbar m\omega}{2}}\, e^{-(-1)^n r}$$

For n even $\quad \Delta x$ increases by a factor e^r

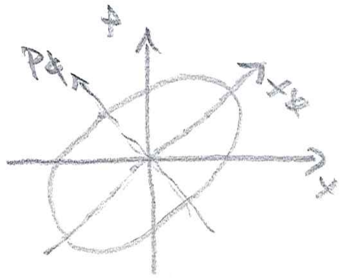
Δp decreases by a factor e^r

For n odd $\quad \Delta x$ decreases and Δp increases.

Spread of wave function in phase space (Wigner function)



e) We guess that for other ϕ ~~the~~ the wavefunction is squeezed in a direction not parallel to the axes. Thus we want to define "rotated" operators x_ϕ and p_ϕ. For this to be meaningful we introduce coordinates with same dimension



$$\xi = x\sqrt{m\omega} = \sqrt{\frac{\hbar}{2}}\,(a^+ + a)$$

$$\pi = \frac{p}{\sqrt{m\omega}} = i\sqrt{\frac{\hbar}{2}}\,(a^+ - a)$$

Coordinates rotated by angle α:

$$\xi_\alpha = \cos\alpha\,\xi - \sin\alpha\,\pi$$

$$\pi_\alpha = \sin\alpha\,\xi + \cos\alpha\,\pi$$

From b): $\langle sq_\zeta|\xi^2|sq_\zeta\rangle = \frac{\hbar}{2}\left[\cosh 2r + \sinh 2r\cos\phi\right]$

$\langle sq_\zeta|\pi^2|sq_\zeta\rangle = \frac{\hbar}{2}\left[\cosh 2r - \sinh 2r\cos\phi\right]$

$$\langle sq_\zeta|\xi_\alpha|sq_\zeta\rangle = \langle sq_\zeta|\pi_\alpha|sq_\zeta\rangle = 0$$

$$\langle sq_\zeta|\xi_\alpha^2|sq_\zeta\rangle = \langle sq_\zeta|\cos^2\alpha\,\xi^2 - \cos\alpha\sin\alpha\,(\xi\pi + \pi\xi) + \sin^2\alpha\,\pi^2|sq_\zeta\rangle$$

We need to find

$$\langle sq_\zeta|\xi\pi|sq_\zeta\rangle = \langle 0|S^+\xi S S^+\pi S|0\rangle$$

$$= i\frac{\hbar}{2}(\cosh r + e^{i\phi}\sinh r)(\cosh r - e^{-i\phi}\sinh r)$$

$$= i\frac{\hbar}{2}\Big[\underbrace{\cosh^2 r - \sinh^2 r}_{1} + \underbrace{\cosh r\sinh r}_{\frac{1}{2}\sinh 2r}\underbrace{(e^{i\phi} - e^{-i\phi})}_{2i\sin\phi}\Big]$$

$$= \frac{\hbar}{2}(i - \sinh 2r\sin\phi) = \langle sq_\zeta|\pi\xi|sq_\zeta\rangle^*$$

$$\Rightarrow \Delta\xi_\alpha^2 = \frac{\hbar}{2}\Big[\cos^2\alpha(\cosh 2r + \sinh 2r\cos\phi) + \sin^2\alpha(\cosh 2r - \sinh 2r\cos\phi)$$
$$+ \cos\alpha \sin\alpha \sinh 2r \sin\phi\Big]$$

$$= \frac{\hbar}{2}\left[\cosh 2r + \sinh 2r\cos(2\alpha - \phi)\right]$$

Similarly we find

$$\Delta\pi_\alpha^2 = \frac{\hbar}{2}\left[\cosh 2r - \sinh 2r\cos(2\alpha - \phi)\right]$$

We reproduce the minimal uncertainty expressions from d) if we choose $2\alpha - \phi = 0 \Rightarrow \alpha = \phi/2$

We should check that the commutator is right.

$$[\xi_\alpha, \pi_\alpha] = [\cos\alpha\,\xi - \sin\alpha\,\pi,\ \sin\alpha\,\xi + \cos\alpha\,\pi]$$

$$= \cos^2\alpha[\xi, \pi] - \sin^2\alpha[\pi, \xi] = [\xi, \pi]$$