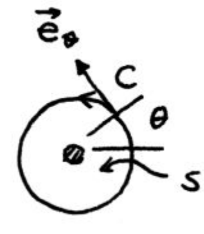
FYS 4110

Midterm exam Oct 2004, Solutions

Problem 1 Particle encircling a magnetic flux

a) Stoke's theorem



$$\phi = \int_S \vec{B}\cdot d\vec{S} = \int_S (\nabla\times\vec{A})\cdot d\vec{S}$$

$$= \oint_C \vec{A}\cdot d\vec{s} = R\int_0^{2\pi} A_\theta\, d\theta$$

Rotational invariance $A_\theta \equiv A$ indep of θ

$$\phi = 2\pi R A \qquad A = \frac{\phi}{2\pi R} \qquad \underline{\vec{A} = \frac{\phi}{2\pi R}\vec{e}_\theta}$$

Outside the solenoid:

$$\vec{B} = \nabla\times\vec{A} = 0$$

$\vec{E} = \vec{B} = 0 \Rightarrow$ no force on the particle

class. eq. of motion not affected by ϕ

b) Momentum operator for particle on the circle:

$$\vec{p} = -i\hbar\nabla \rightarrow \cancel{-i\frac{\hbar}{R}\frac{\partial}{\partial\theta}} \; -i\frac{\hbar}{R}\vec{e}_\theta\frac{\partial}{\partial\theta}$$

when acting on wave functions $\psi(\theta)$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \rightarrow \frac{1}{2m}\left(-i\frac{\hbar}{R}\frac{\partial}{\partial\theta} - \frac{e}{c}A\right)^2$$

$$= -\frac{\hbar^2}{2mR^2}\left(\frac{\partial}{\partial\theta} - i\frac{e\phi}{2\pi\hbar c}\right)^2$$

$$= \underline{-\frac{\hbar^2}{2mR^2}\left(\frac{\partial}{\partial\theta} - i\varkappa\right)^2} \qquad \varkappa \equiv \frac{e\phi}{2\pi\hbar c}$$

angular mom. eigenstates

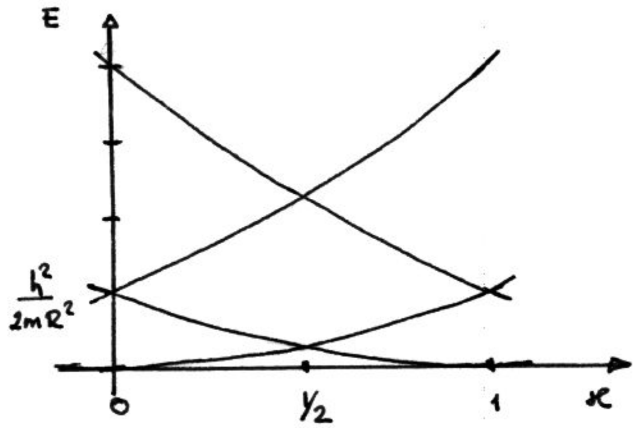
= energy eigenstates

Angular momentum eigenstates

$$-i\hbar\frac{\partial}{\partial\theta}\psi = l\hbar\psi \Rightarrow \psi_l(\theta) = \frac{1}{\sqrt{2\pi}}e^{il\theta}$$

$$H\psi_l = -\frac{\hbar^2}{2mR^2}(il - i\varkappa)^2\psi_l$$

$$\Rightarrow \underline{E_l = \frac{\hbar^2}{2mR^2}(l-\varkappa)^2}$$



Periodic variation :

$\varkappa \to \varkappa + 1$ and $l \to l+1$ leaves the energy unchanged

$\Rightarrow$ the set of energies $\{E_l, l = 0, \pm 1, \dots\}$ is invariant

when $\varkappa \to \varkappa + 1$

Expressed in terms of the flux :

$$\frac{e\phi}{2\pi\hbar c} \to \frac{e\phi}{2\pi\hbar c} + 1 \Rightarrow \phi \to \phi + \frac{2\pi\hbar c}{e}, \quad \underline{\phi_0 = \frac{2\pi\hbar c}{e}} \text{ flux quantum}$$

Angular momentum of ground state :

$0 \leq \phi < \frac{\phi_0}{2}$: $l = 0$

$\frac{\phi_0}{2} < \phi \leq \phi_0$: $l = 1$

$\phi = \frac{\phi_0}{2}$: Spectrum is doubly degenerate

Ground state : $l = 0$ and $l = 1$ same energy.

c) Probability current

$$J = -\frac{i\hbar}{2mR}(\psi^*\partial_\theta\psi - \psi\partial_\theta\psi^*) - \frac{e\phi}{2\pi Rmc}\psi^*\psi \qquad \partial_\theta = \frac{\partial}{\partial\theta}$$

in ang. mom state l ; $\psi = \psi_l$ $\quad \partial_\theta \psi_l = il\psi_l$

$$\Rightarrow \quad J_l = \left(\frac{\hbar}{mR}l - \frac{e\phi}{2\pi Rmc}\right)\frac{1}{2\pi}$$

$$= \frac{\hbar}{2\pi mR}(l - \varkappa)$$

Ground state

$0 \leq \phi < \frac{\phi_0}{2}$ $(l=0)$ $\quad \underline{J_0 = -\frac{\varkappa\hbar}{2\pi mR}}$

$\frac{\phi_0}{2} < \phi \leq \phi_0$ $(l=1)$ $\quad \underline{J_1 = \frac{(1-\varkappa)\hbar}{2\pi mR}}$

Maximum value $(\phi = \frac{\phi_0}{2})$:

$$\underline{J_0 = -\frac{\hbar}{4\pi mR}} \qquad \underline{J_1 = \frac{\hbar}{4\pi mR}}$$

two possible values due to degeneracy

Velocity : $\quad J = \rho v \qquad \rho = \psi^*\psi = \frac{1}{2\pi}$

$$\Rightarrow \quad v_0 = 2\pi J_0 = \underline{-\frac{\hbar}{2mR}} \qquad (l=0)$$

$$v_1 = 2\pi J_1 = \underline{\frac{\hbar}{2mR}} \qquad (l=1)$$

d) Propagator

$$G(\theta,t;0,0) = \langle\theta| e^{-\frac{i}{\hbar}Ht}|0\rangle$$

$$= \sum_l \langle\theta| e^{-\frac{i}{\hbar}Ht}|l\rangle\langle l|0\rangle$$

$$= \sum_l e^{-\frac{i}{\hbar}E_l t}\langle\theta|l\rangle\langle l|0\rangle$$

$$\langle\theta|l\rangle = \psi_l(\theta) = \frac{1}{\sqrt{2\pi}} e^{il\theta}$$

$$\langle l|0\rangle = \psi_l(0)^* = \frac{1}{\sqrt{2\pi}}$$

$$G(\theta t;0,0) = \frac{1}{2\pi}\sum_{l=-\infty}^{+\infty} \exp\Big\{\underbrace{-\frac{i}{\hbar}\frac{\hbar^2}{2mR^2}(l-\varkappa)^2 t + il\theta}\Big\}$$

$$= -i\frac{\hbar}{2mR^2}(l^2-2l\varkappa+\varkappa^2)t + il\theta$$

$$= i\Big\{-\frac{\hbar t}{2mR^2}l^2 + \Big(\theta + \frac{\varkappa\hbar t}{mR^2}\Big)l\Big\} - i\frac{\hbar t\varkappa^2}{2mR^2}$$

$$\equiv i\{\pi w l^2 + 2zl\} - i\frac{\hbar t\varkappa^2}{2mR^2}$$

$$G(\theta t,0,0) = \frac{1}{2\pi}\exp\Big\{-i\frac{\varkappa^2\hbar}{2mR^2}t\Big\}\sum_{l=-\infty}^{+\infty}\exp\{i[\pi w l^2 + 2zl]\}$$

$$= \frac{1}{2\pi}\exp\Big\{-i\frac{\varkappa^2\hbar}{2mR^2}t\Big\}\,\vartheta_3(z,w)$$

$$= \underline{\frac{1}{2\pi}\exp\Big\{-i\frac{\varkappa^2\hbar}{2mR^2}t\Big\}\,\vartheta_3\Big(\frac{1}{2}\Big(\theta + \frac{\varkappa\hbar t}{mR^2}\Big), -\frac{\hbar t}{2\pi mR^2}\Big)}$$

e) Classical paths

$$\theta(t') = \frac{\theta + 2\pi n}{t}t' \qquad n = 0, \pm 1, \dots$$

$$\dot\theta(t') = \frac{\theta+2\pi n}{t} \qquad (\text{const})$$

Action

$$S = \frac{1}{2}mv^2 t + \frac{e}{c}Av\,t \qquad \frac{e}{c}Av = \frac{e}{c}\frac{\phi}{2\pi R}R\dot\theta$$

$$= \underbrace{\frac{e\phi}{2\pi c}}_{=\hbar\varkappa}\dot\theta$$

$$S_n = \frac{1}{2}mR^2\frac{(\theta+2\pi n)^2}{t} + \frac{e\phi}{2\pi c}(\theta+2\pi n)$$

$$= \frac{1}{2}mR^2\frac{4\pi^2}{t}n^2 + \frac{1}{2}mR^2\frac{4\pi\theta}{t}n + 2\pi\hbar\varkappa n$$

$$+ \frac{1}{2}mR^2\frac{\theta^2}{t} + \frac{e\phi}{2\pi c}\theta$$

$$S_n = 2\pi^2 \frac{mR^2}{t} n^2 + \frac{2\pi mR^2}{t}\left(\theta + \frac{\varkappa\hbar t}{mR^2}\right) n + \frac{1}{2}\frac{mR^2}{t}\left(\theta^2 + 2\frac{\varkappa\hbar t}{mR^2}\theta\right)$$

define $\bar\theta = \theta + \frac{\varkappa\hbar t}{mR^2}$

$$S_n = 2\pi^2 \frac{mR^2}{t} n^2 + \frac{2\pi mR^2}{t}\bar\theta\, n + \frac{1}{2}\frac{mR^2}{t}\bar\theta^2 - \frac{1}{2}\frac{\varkappa^2\hbar^2}{mR^2} t$$

Path integral representation

$$G(\theta t, 0 0) = N \sum_{n=-\infty}^{+\infty} \exp\left\{\frac{i}{\hbar} S_n\right\} \qquad N = \sqrt{\frac{mR^2}{2\pi i\hbar t}} \text{ (prob. 2.41}$$

$$= N \exp\left\{\frac{i}{2}\left(\frac{mR^2}{\hbar t}\bar\theta^2 - \frac{\varkappa^2\hbar}{mR^2}t\right)\right\}$$

$$\times \sum_{n=-\infty}^{+\infty} \exp\Big\{ i \underbrace{\left(2\pi^2 \frac{mR^2}{\hbar t} n^2 + \frac{2\pi mR^2}{\hbar t}\bar\theta\, n\right)}_{\equiv\, \pi\omega' n^2 + 2z' n}\Big\}$$

$$= N \exp\left\{\frac{i}{2}\left(\frac{mR^2}{\hbar t}\bar\theta^2 - \frac{\varkappa^2\hbar}{mR^2}t\right)\right\} \vartheta_3(z', \omega')$$

$$= \quad \text{——} \quad '' \quad \text{——} \quad \vartheta_3\left(\frac{\pi mR^2}{\hbar t}\bar\theta,\ 2\pi\frac{mR^2}{\hbar t}\right)$$

$$= \sqrt{\frac{mR^2}{2\pi i\hbar t}}\ \exp\left\{\frac{i}{2}\left(\frac{mR^2}{\hbar t}\bar\theta^2 - \frac{\varkappa^2\hbar}{mR^2}t\right)\right\} \vartheta_3\left(\frac{\pi mR^2}{\hbar t}\bar\theta,\ 2\pi\frac{mR^2}{\hbar t}\right)$$

Apply relation

$$\vartheta_3(z', \omega') = (-i\omega')^{-1/2} e^{z'^2/i\pi\omega'}\, \vartheta_3\left(\frac{z'}{\omega'}, -\frac{1}{\omega'}\right)$$

$$\frac{z'}{\omega'} = \frac{\pi mR^2}{\hbar t}\bar\theta\ \frac{\hbar t}{2\pi mR^2} = \frac{1}{2}\bar\theta$$

$$-\frac{1}{\omega'} = -\frac{\hbar t}{2\pi mR^2}$$

Inserted :

$$G(\theta t, 0 0) = \sqrt{\frac{mR^2}{2\pi i \hbar t}} \sqrt{\frac{i\hbar t}{2\pi m R^2}} \exp\left\{ \frac{i}{2} \left(\frac{mR^2}{\hbar t} \bar{\theta}^2 - \frac{\varkappa^2 \hbar}{mR^2} \right) t \right\}$$

$$\times \exp\left\{ -\frac{i}{2} \frac{mR^2}{\hbar t} \bar{\theta}^2 \right\} \vartheta_3\left(\tfrac{1}{2}\bar{\theta}, -\frac{\hbar t}{2\pi m R^2} \right)$$

$$= \frac{1}{2\pi} \exp\left\{ -i \frac{\varkappa \hbar}{2mR^2} t \right\} \vartheta_3\left(\tfrac{1}{2}\left(\theta + \frac{\varkappa \hbar t}{mR^2}\right), -\frac{\hbar t}{2\pi m R^2} \right)$$

same as obtained by direct calculation.

Problem 2 Entangled photons

a) $\frac{n_1}{N}$ approach P_1 for larg N

$\frac{n_2}{N}$ — " — P_2 — " —

$\frac{n_{12}}{N}$ — " — P_{12} — " —

b) Density operator

$$|H\rangle_1 \otimes |V\rangle_2$$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\big(|HV\rangle\langle HV| + |VH\rangle\langle VH|$$

$$+ e^{i\chi} |VH\rangle\langle HV| + e^{-i\chi} |HV\rangle\langle VH| \big\}$$

$$\rho_1 = \mathrm{Tr}_2\, \rho = {}_2\langle H| \rho |H\rangle_2 + {}_2\langle V| \rho |V\rangle_2$$

$$= \frac{1}{2}\big(|H\rangle\langle H| + |V\rangle\langle V| \big)_1$$

$$= \underline{\frac{1}{2} \mathbb{1}_1}$$

$$\rho_2 = \mathrm{Tr}_1\, \rho = \underline{\frac{1}{2} \mathbb{1}_2}$$

c) $\chi = \pi$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$$

$$\rho = \frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH| - |VH\rangle\langle HV| - |HV\rangle\langle VH|)$$

$$P_1 = Tr(\rho P_1) = Tr_1(\rho_1 P_1) = \frac{1}{2} Tr P_1 = \frac{1}{2}\langle\theta_1|\theta_1\rangle = \underline{\frac{1}{2}}$$

$$P_2 = Tr_2(\rho_2 P_2) = \underline{\frac{1}{2}}$$

$$P_{12} = Tr(\rho P_{12}) = Tr(\rho\, |\theta_1\theta_2\rangle\langle\theta_1\theta_2|)$$

$$= \langle\theta_1\theta_2|\rho|\theta_1\theta_2\rangle$$

$$= \frac{1}{2}\{\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2 - 2\cos\theta_1 \sin\theta_1 \cos\theta_2 \sin\theta_2\}$$

$$= \frac{1}{2}(\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2)^2$$

$$= \underline{\frac{1}{2}\sin^2(\theta_1 - \theta_2)}$$

$$P_{12} = \langle P_1 P_2\rangle = \frac{1}{2}\sin^2(\theta_1 - \theta_2)$$

$$\langle P_1\rangle\langle P_2\rangle = \frac{1}{4}$$

$P_{12} \neq \langle P_1\rangle\langle P_2\rangle$ unless $\sin(\theta_1 - \theta_2) = \frac{1}{\sqrt{2}}$

shows correlations

d) $\chi = 0$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)$$

$$\rho = \frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH| + |HV\rangle\langle VH| + |VH\rangle\langle HV|)$$

$P_1 = Tr_1(\rho_1 P_1) = \underline{\frac{1}{2}}$ $\quad P_2 = Tr_2(\rho_2 P_2) = \underline{\frac{1}{2}}$ as before

$$P_{12} = \langle \theta_1 \theta_2 | \rho | \theta_1 \theta_2 \rangle$$

$$= \frac{1}{2} \left(\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2 + 2\cos\theta_1 \sin\theta_1 \cos\theta_2 \sin\theta_2 \right)$$

$$= \frac{1}{2} \left(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 \right)^2$$

$$= \underline{\frac{1}{2} \sin^2(\theta_1 + \theta_2)}$$

c) is rotationally invariant:

when $\theta_1 \to \theta_1 + \alpha$, $\theta_2 \to \theta_2 + \alpha$

d) is not, but invariant when $\theta_1 \to \theta_1 + \alpha$, $\theta_2 \to \theta_2 - \alpha$

e) $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|HV\rangle + i\, |VH\rangle \right)$

$$\rho = \frac{1}{2} \left(|HV\rangle\langle HV| + |VH\rangle\langle VH| + i \left(|VH\rangle\langle HV| - |HV\rangle\langle VH| \right) \right)$$

$\underline{P_1 = \frac{1}{2}}$, $\underline{P_2 = \frac{1}{2}}$ as before

$$P_{12} = \langle \theta_1 \theta_2 | \rho | \theta_1 \theta_2 \rangle$$

$$= \frac{1}{2} \left(\cos^2\theta_1 \sin^2\theta_2 + \sin^2\theta_1 \cos^2\theta_2 \right)$$

$$= \underline{\frac{1}{4} \left(\sin^2(\theta_1 - \theta_2) + \sin^2(\theta_1 + \theta_2) \right)}$$

No contributions from mixed terms $|VH\rangle\langle HV|$, $|HV\rangle\langle VH|$,

Same result as with

$$\rho = \underline{\frac{1}{2} \left(|HV\rangle\langle HV| + |VH\rangle\langle VH| \right)}$$

incoherent mixture (mixed state) of $|HV\rangle$ and $|VH\rangle$

f) Bell inequality

$$F(0,\theta,2\theta) = P_{12}(\theta,2\theta) - |P_{12}(0,\theta) - P_{12}(0,2\theta)|$$

case I :

$$P_{12}(\theta_1,\theta_2) = \frac{1}{2}\sin^2(\theta_1-\theta_2)$$

$$F_I(0,\theta,2\theta) = \frac{1}{2}\{\sin^2\theta - |\sin^2\theta - \sin^2 2\theta|\}$$

case II

$$P_{12}(\theta_1,\theta_2) = \frac{1}{2}\sin^2(\theta_1+\theta_2)$$

$$F_{II}(0,\theta,2\theta) = \frac{1}{2}\{\sin^2 3\theta - |\sin^2\theta - \sin^2 2\theta|\}$$

case III

$$P_{12}(\theta_1,\theta_2) = \frac{1}{4}\left(\sin^2(\theta_1+\theta_2) + \sin^2(\theta_1-\theta_2)\right)$$

$$F_{III}(0,\theta,3\theta) = \frac{1}{4}\{\sin^2 3\theta + \sin^2\theta - 2|\sin^2\theta - \sin^2 2\theta|$$

Plot shows

Condition $F(0,\theta,2\theta) \geq 0$

is not satisfied for I and II,

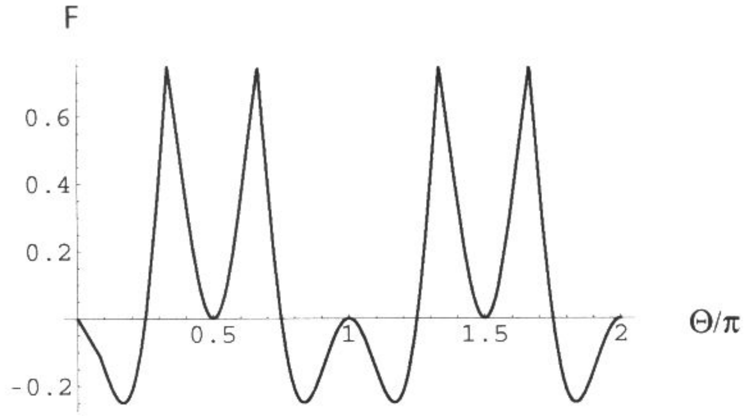
but is satisfied for III

F_{III} is the same as for the non-entangled

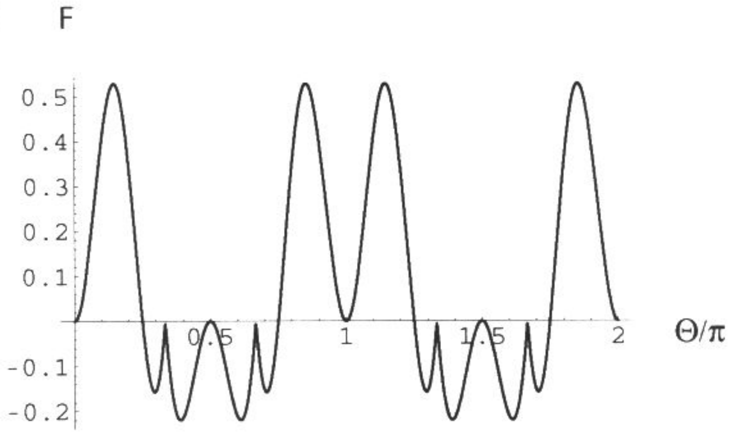
mixed state $\frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH|)$,

should <u>not</u> show breaking of the Bell inequality.

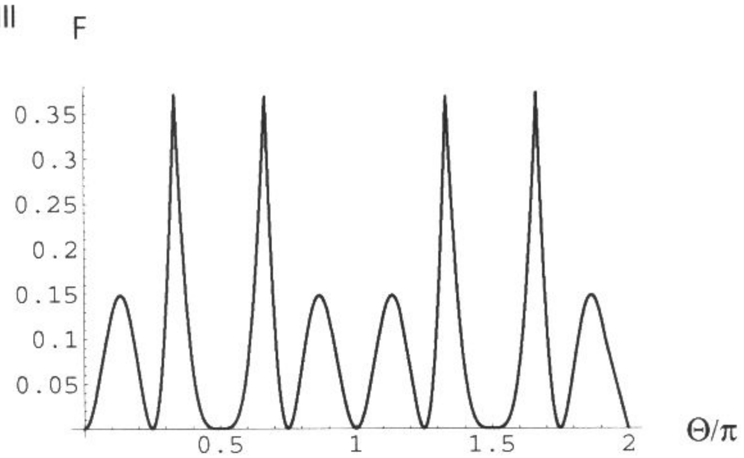
Case I



Case II



Case III



Midterm Exam 2005 Solutions

Problem 1 Spin motion in an oscillating field

a) Time evolution of density matrix

$$\rho(t) = U(t)\,\rho_0\,U^+(t)$$

In magnetic field

$$H = -\vec{\mu}\cdot\vec{S} = -\frac{eB}{2mc}\hbar\,\sigma_z$$

$$= \frac{1}{2}\hbar\omega_0\sigma_z \qquad \omega_0 = -\frac{eB}{mc}$$

$$\Rightarrow U(t) = e^{-\frac{i}{\hbar}Ht} = e^{-\frac{i}{2}\omega_0\sigma_z t}$$

$$\vec{r}(t)\cdot\vec{\sigma} = \vec{r}_0\cdot U(t)\,\vec{\sigma}\,U^+(t)$$

$$U\sigma_z U^+ = \sigma_z$$

$$U\sigma_x U^+ = e^{-\frac{i}{2}\omega_0\sigma_z t}\,\sigma_x\,e^{\frac{i}{2}\omega_0\sigma_z t}$$

$$= \sigma_x - \frac{i}{2}\omega_0 t\,[\sigma_z,\sigma_x] + \frac{1}{2!}\left(-\frac{i}{2}\omega_0 t\right)^2[\sigma_z,[\sigma_z,\sigma_x]] + \cdots$$

$$[\sigma_z,\sigma_x] = 2i\,\sigma_y$$

$$[\sigma_z,\sigma_y] = -2i\,\sigma_x$$

$$\Rightarrow U\sigma_x U^+ = \sigma_x + \omega_0 t\,\sigma_y - \frac{1}{2}(\omega_0 t)^2\sigma_x - \frac{1}{3!}(\omega_0 t)^3\sigma_y + \cdots$$

$$= \sigma_x\cos\omega_0 t + \sigma_y\sin\omega_0 t$$

$$U\sigma_y U^+ = \sigma_y - \omega_0 t\,\sigma_x - \frac{1}{2}(\omega_0 t)^2\sigma_y + \frac{1}{3!}(\omega_0 t)^3\sigma_y - \cdots$$

$$= -\sigma_x\sin\omega_0 t + \sigma_y\cos\omega_0 t$$

$$\Rightarrow \vec{r}(t)\cdot\vec{\sigma} = (x_0\cos\omega_0 t - y_0\sin\omega_0 t)\,\sigma_x + (x_0\sin\omega_0 t + y_0\cos\omega_0 t)\,\sigma_y + \sigma_z z_0$$

$$\Rightarrow \underline{\vec{r}(t) = (x_0\cos\omega_0 t - y_0\sin\omega_0 t)\,\vec{i} + (x_0\sin\omega_0 t + y_0\cos\omega_0 t)\,\vec{j} + z_0\vec{k}}$$

$\vec{r}$ rotates with angular velocity ω_0 around the z-axis

b) Initial condition $\vec{r}_0 = a\vec{k}$

$$\Rightarrow \quad \rho_0 = \frac{1}{2}(\mathbb{1} + a\sigma_z)$$

$$= \frac{1}{2}\begin{pmatrix} 1+a & 0 \\ 0 & 1-a \end{pmatrix}$$

positivity: $\left.\begin{array}{l} 1+a \geq 0 \Rightarrow a \geq -1 \\ 1-a \geq 0 \Rightarrow a \leq 1 \end{array}\right\} -1 \leq a \leq 1$

Time evolution operator with oscillating field (sect. 1.3.2)

$$U(t) = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}$$

$$A = \left(\cos\frac{\Omega t}{2} - i\cos\theta\sin\frac{\Omega t}{2}\right) e^{-\frac{i}{2}\omega t}$$

$$B = -i\sin\theta\sin\frac{\Omega t}{2}\, e^{-\frac{i}{2}\omega t}$$

$$\cos\theta = \frac{\omega_0 - \omega}{\sqrt{(\omega_0-\omega)^2 + \omega_1^2}} \qquad \sin\theta = \frac{\omega_1}{\sqrt{(\omega_0-\omega)^2+\omega_1^2}}$$

$$\omega_1 = -\frac{eB_1}{mc} \qquad \Omega = \sqrt{(\omega_0-\omega)^2 + \omega_1^2}$$

Time evolution

$$\vec{r}(t)\cdot\vec{\sigma} = \vec{r}_0 \cdot U(t)\vec{\sigma}\, U^\dagger(t)$$

$$= a\, U(t)\sigma_z U^\dagger(t)$$

$$= a\begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A^* & -B \\ B^* & A \end{pmatrix}$$

$$= a\begin{pmatrix} |A|^2 - |B|^2 & -2AB \\ -2A^*B^* & -(|A|^2-|B|^2) \end{pmatrix}$$

$$z(t) = a(|A|^2 - |B|^2) = a\left(\cos^2\frac{\Omega t}{2} + \cos^2\theta\sin^2\frac{\Omega t}{2} - \sin^2\theta\sin^2\frac{\Omega t}{2}\right)$$

$$= \frac{a}{2}\left((1+\cos\Omega t) + (\cos^2\theta - \sin^2\theta)(1-\cos\Omega t)\right)$$

$$\underline{= a\left(\cos^2\theta + \sin^2\theta\cos\Omega t\right)}$$

$$x(t) - iy(t) = -2a\,AB$$

$$= 2a\left[\cos\theta\sin\theta\sin^2\frac{\Omega t}{2} + i\sin\theta\cos\frac{\Omega t}{2}\sin\frac{\Omega t}{2}\right]e^{-i\omega t}$$

$$= 2a\sin\theta\sin\frac{\Omega t}{2}\left[\left(\cos\theta\sin\frac{\Omega t}{2}\cos\omega t + \cos\frac{\Omega t}{2}\sin\omega t\right)\right.$$

$$\left. + i\left(\cos\frac{\Omega t}{2}\cos\omega t - \cos\theta\sin\frac{\Omega t}{2}\sin\omega t\right)\right]$$

$\Rightarrow$

$$\underline{x(t) = 2a\sin\theta\sin\frac{\Omega t}{2}\left(\cos\frac{\Omega t}{2}\sin\omega t + \cos\theta\sin\frac{\Omega t}{2}\cos\omega t\right)}$$

$$\underline{y(t) = -2a\sin\theta\sin\frac{\Omega t}{2}\left(\cos\frac{\Omega t}{2}\cos\omega t - \cos\theta\sin\frac{\Omega t}{2}\sin\omega t\right)}$$

c) Resonance : $\omega = \omega_0$

$\Rightarrow \Omega = \omega_1$, $\cos\theta = 0$, $\sin\theta = 1$

$$\Rightarrow \quad z(t) = a\cos\omega_1 t$$

$$x(t) = a\sin\omega_1 t\sin\omega t$$

$$y(t) = -a\sin\omega_1 t\cos\omega t$$

Oscillations in the z coordinate combined with rotation about the z-axis

Problem 2 Charged particle in a strong magnetic field

a) $m\vec{a} = \frac{e}{c}\vec{v}\times\vec{B}$

$$\Rightarrow \dot{\vec{v}} = \frac{eB}{mc}\vec{v}\times\vec{k} = \vec{\omega}\times\vec{v} \qquad \vec{\omega} = -\frac{eB}{mc}\vec{k}$$

$$\Rightarrow \dot{\vec{r}} = \vec{\omega}\times\vec{r} + \vec{C} \text{ (const.)}$$

$$\equiv \underline{\vec{\omega}\times(\vec{r}-\vec{r}_0)} \qquad \vec{C} = -\vec{\omega}\times\vec{r}_0$$

Circular motion with angular velocity about a point $\vec{r}_0$.

$$\frac{d}{dt}[m\vec{r}\times\vec{v}] = m\vec{r}\times\vec{a}$$

$$= \vec{r}\times(\tfrac{e}{c}\vec{v}\times\vec{B})$$

$$= -\tfrac{e}{c}\vec{r}\cdot\vec{v}\,\vec{B} \qquad (\vec{r}\cdot\vec{B}=0)$$

$$= \frac{d}{dt}\left(-\frac{e\vec{B}}{2c}r^2\right)$$

$$\Rightarrow \frac{d}{dt}L_{mek} = -\frac{d}{dt}\left(\frac{eB}{2c}r^2\right) \quad \text{generally different from } 0$$

conserved only when $r = \text{const}$ $(r_0 = 0)$

$$\frac{d}{dt}L = \underline{\frac{d}{dt}\left(L_m + \frac{eB}{2c}r^2\right) = 0} \quad \text{always conserved}$$

b) $\vec{R} = \vec{r} + \frac{1}{\omega}\vec{k}\times\vec{v}$

$$\Rightarrow \dot{\vec{R}} = \vec{v} + \frac{1}{\omega}\vec{k}\times\vec{a}$$

$$= \vec{v} + \frac{1}{m\omega}\frac{e}{c}\vec{k}\times(\vec{v}\times\vec{B})$$

$$= \vec{v} + \frac{1}{\omega}\frac{eB}{mc}\vec{v} \qquad (\vec{k}\cdot\vec{v}=0,\ \omega = -\frac{eB}{mc})$$

$$= \underline{0}$$

Circular orbits

$$\vec{v} = \vec{\omega} \times (\vec{r} - \vec{r}_0)$$

$$\vec{k} \times \vec{v} = \vec{k} \times (\vec{k} \times (\vec{r} - \vec{r}_0))\,\omega$$

$$= -\omega(\vec{r} - \vec{r}_0)$$

$$\Rightarrow \vec{R} = \vec{r} + \frac{1}{\omega}(\vec{k} \times \vec{v}) = \vec{r} - (\vec{r} - \vec{r}_0) = \underline{\vec{r}_0}$$

$$\vec{\rho} = \frac{1}{\omega}\,\vec{k} \times \vec{v} = \underline{\vec{r}_0 - \vec{r}}$$

$\vec{R}$ = center of orbit

$\vec{\rho}$ = vector from particle to center of orbit.

c) $$m\vec{v} = \vec{p} - \frac{e}{c}\vec{A} = \vec{p} + \frac{e}{2c}\,\vec{r} \times \vec{B} = \vec{p} + \frac{eB}{2c}\,\vec{r} \times \vec{k}$$

$$\vec{R} = \vec{r} + \frac{1}{\omega}\vec{k} \times \vec{v}$$

$$= \vec{r} + \frac{1}{m\omega}\,\vec{k} \times (\vec{p} - \frac{e}{c}\vec{A})$$

$$= \vec{r} + \underbrace{\frac{1}{m\omega}\,\frac{eB}{2c}}_{-\frac{1}{2}}\,\underbrace{\vec{k} \times (\vec{r} \times \vec{k})}_{\vec{r}} + \frac{1}{m\omega}\,\vec{k} \times \vec{p}$$

$$= \underline{\frac{1}{2}\vec{r} + \frac{1}{m\omega}\,\vec{k} \times \vec{p}}$$

$$\hat{X} = \frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y \quad , \quad \hat{Y} = \frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x$$

$$[\hat{X}, \hat{Y}] = \frac{1}{2m\omega}\left([\hat{x}, \hat{p}_x] - [\hat{p}_y, \hat{y}]\right)$$

$$= i\,\frac{\hbar}{m\omega} = i\,\frac{\hbar c}{|eB|} = \underline{i\,l_B^2} \qquad l_B = \sqrt{\frac{\hbar c}{|eB|}}$$

$$\vec{\rho} = \vec{R} - \vec{r}$$

$$\Rightarrow \hat{\rho}_x = -\left(\frac{1}{2}\hat{x} + \frac{1}{m\omega}\hat{p}_y\right), \quad \hat{\rho}_y = -\left(\frac{1}{2}\hat{y} - \frac{1}{m\omega}\hat{p}_x\right)$$

$$\Rightarrow [\hat{\rho}_x, \hat{\rho}_y] = \frac{1}{2m\omega}\left(-[\hat{x}, \hat{p}_x] + [\hat{p}_y, \hat{y}]\right) = \underline{-i l_B^2}$$

$(\hat{X}, \hat{Y})$ commutes like phase space variables $(\hat{X}, \hat{P})$

with $\hat{Y} = \frac{1}{m\omega}\hat{P}$

d) $\hat{a} = \frac{1}{\sqrt{2}\,l_B}(\hat{X} + i\hat{Y})$, $\hat{b} = \frac{1}{\sqrt{2}\,l_B}(\hat{\rho}_x - i\hat{\rho}_y)$

$$\Rightarrow \quad [\hat{a}, \hat{a}^+] = \frac{1}{2l_B^2}\left(-i[\hat{X},\hat{Y}] + i[\hat{Y},\hat{X}]\right) = 1$$

$$[\hat{b}, \hat{b}^+] = \frac{1}{2l_B^2}\left(i[\hat{\rho}_x,\hat{\rho}_y] - i[\hat{\rho}_y,\hat{\rho}_x]\right) = 1$$

$$[\hat{a}, \hat{b}] = \frac{1}{2l_B^2}\left([\hat{X},\hat{\rho}_x] + [\hat{Y},\hat{\rho}_y] + i\left([\hat{Y},\hat{\rho}_x] - [\hat{X},\hat{\rho}_y]\right)\right)$$

$$[\hat{X},\hat{\rho}_x] = [\hat{Y},\hat{\rho}_y] = 0$$

$$[\hat{Y},\hat{\rho}_x] = \left[\tfrac{1}{2}\hat{y} + \tfrac{1}{m\omega}\hat{p}_x , -\left(\tfrac{1}{2}\hat{x} + \tfrac{1}{m\omega}\hat{p}_y\right)\right]$$

$$= -\frac{1}{2m\omega}\left([\hat{y},\hat{p}_y] + [\hat{p}_x,\hat{x}]\right) = 0$$

$[\hat{X},\hat{\rho}_y] = 0$ similary

$$\Rightarrow \quad [\hat{a},\hat{b}] = 0$$

$[\hat{a},\hat{b}^+] = 0$ similar calculations

$(\hat{a},\hat{a}^+)$, $(b,\hat{b}^+)$ commut. relations as for two indep. harm. osc.

e) $H = \frac{1}{2}mv^2$, $\vec{v} = \vec{\omega}\times(\vec{r}-\vec{R})$ $\Rightarrow v^2 = \omega^2(\vec{r}-\vec{R})^2 = \omega^2\vec{\rho}^{\,2}$

$$\hat{H} = \tfrac{1}{2}m\omega^2(\hat{\rho}_x^2 + \hat{\rho}_y^2) \qquad \hat{\rho}_x = \frac{l_B}{\sqrt{2}}(b + b^+)$$

$$= \tfrac{1}{2}m\omega^2 l_B^2(bb^+ + b^+b) \qquad \hat{\rho}_y = i\frac{l_B}{\sqrt{2}}(b - b^+)$$

$$= \underline{\hbar\omega\left(b^+b + \tfrac{1}{2}\right)}$$ harm osc. spectrum $\underline{E_n = \hbar\omega(n + \tfrac{1}{2})}$

Note energy spectrum independent of m

$$L = (m\vec{r}\times\vec{v})_z + \frac{eB}{2c}r^2 \qquad \vec{r} = \vec{R}-\vec{\rho}\ ,\ \vec{v} = -\vec{\omega}\times\vec{\rho}$$

$$\vec{r}\times\vec{v} = -(\vec{R}-\vec{\rho})\times(\vec{\omega}\times\vec{\rho})$$

$$= \omega(\rho^2 - \vec{R}\cdot\vec{\rho})$$

$$r^2 = R^2 + \rho^2 - 2\vec{R}\cdot\vec{\rho}$$

$$\hat{L} = m\omega\left((\hat{\rho}^2 - \hat{\vec{R}}\cdot\hat{\vec{\rho}}) - \tfrac{1}{2}(\hat{R}^2 + \hat{\rho}^2 - 2\hat{\vec{R}}\cdot\hat{\vec{\rho}})\right)$$

$$= \tfrac{1}{2}m\omega(\hat{\rho}^2 - \hat{R}^2)$$

$$\underline{= \hbar(b^\dagger b - a^\dagger a)}$$

Ground state = lowest Landau level:

$n=0 \Rightarrow E_0 = \frac{1}{2}\hbar\omega$, no restriction on m

$|m\rangle = |m,0\rangle$ $\quad m = 0,1,2\ldots$ orthonorm. basis in the lowest Landau level

Coherent state

$$\hat{a}|z\rangle = z|z\rangle\ ,\quad \hat{b}|z\rangle = 0$$

$$\hat{x} = \hat{X} - \hat{\rho}_x = \frac{\ell_B}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger + \hat{b} + \hat{b}^\dagger)$$

$$\hat{y} = \hat{Y} - \hat{\rho}_y = i\frac{\ell_B}{\sqrt{2}}(-\hat{a} + \hat{a}^\dagger + \hat{b} - \hat{b}^\dagger)$$

$$\langle z|\hat{x}|z\rangle = \langle z|\hat{X}|z\rangle = \frac{\ell_B}{\sqrt{2}}(z+z^*) = \underline{\sqrt{2}\,\ell_B\,\mathrm{Re}\,z}$$

$$\langle z|\hat{y}|z\rangle = \langle z|\hat{Y}|z\rangle = -i\frac{\ell_B}{\sqrt{2}}(z-z^*) = \underline{\sqrt{2}\,\ell_B\,\mathrm{Im}\,z}$$

Expanded in $|m\rangle$-states

$$|z\rangle = \sum_m c_m|m\rangle \qquad a|z\rangle = \sum_m c_m\sqrt{m}\,|m-1\rangle$$

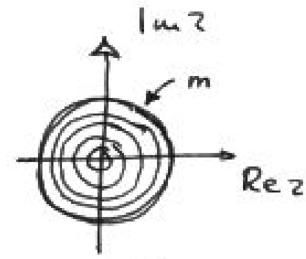
$$z|z\rangle = \sum_m z\,c_{m-1}\,|m-1\rangle$$

$$\Rightarrow c_m = \frac{c_{m-1}}{\sqrt{m}}z = \frac{c_{m-2}}{\sqrt{m(m-1)}}z^2 = \ldots = \frac{c_0}{\sqrt{m!}}z^m\,|m\rangle$$

Normalization $\quad 1 = \langle z|z\rangle = |c_0|^2\sum_m\frac{|z|^{2m}}{m!} = |c_0|^2 e^{-|z|^2}$

$$|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_m \frac{z^m}{\sqrt{m!}} |m\rangle$$

$$|\langle m|z\rangle|^2 = \frac{|z|^{2m}}{m!} e^{-|z|^2}$$



In the z-plane: maximum around a circle of radius $|z|^2 = m$

In the x, y plane: maximum at $r^2 = 2m\ell_B^2$

Area within state m:

$$A_m = \pi r_m^2 = 2\pi m \ell_B^2 \quad \text{increases linearly with } m$$

Number of states = m

$$\Rightarrow \text{density of states } \sigma = \frac{m}{A_m} = \underline{\frac{1}{2\pi\ell_B^2}}$$

g) $\hat{H} = \hat{H}_0 - eE\hat{x}$ $\qquad H_0 = \hbar\omega(b^\dagger b + \frac{1}{2})$

in the lowest Landau level $H_0 \to \frac{1}{2}\hbar\omega$, $\hat{x} \to \hat{X}$

$$\Rightarrow \underline{\hat{H} = \frac{1}{2}\hbar\omega - \frac{1}{\sqrt{2}}\ell_B eE(a + a^\dagger)}$$

Time evolution

$$U(t) = \exp\{-\frac{i}{\hbar}\hat{H}t\} = e^{-\frac{i}{2}\omega t} e^{i\frac{\ell_B}{\sqrt{2}\hbar}eE(a+a^\dagger)t}$$

$$a(t) = U^\dagger(t)\, a\, U(t) = e^{-i\frac{\ell_B}{\sqrt{2}\hbar}eE(a+a^\dagger)t}\, a\, e^{i\frac{\ell_B}{\sqrt{2}\hbar}eE(a+a^\dagger)t}$$

$$= a - i\frac{\ell_B}{\sqrt{2}\hbar}eE[a+a^\dagger, a]\,t$$

$$= a + i\frac{\ell_B}{\sqrt{2}\hbar}eEt$$

$$a^\dagger(t) = a^\dagger - i\frac{\ell_B}{\sqrt{2}\hbar}eEt$$

Heisenberg picture

$$\hat{X}(t) = \frac{l_B}{\sqrt{2}}\left(\hat{a}(t) + \hat{a}^{+}(t)\right) = \frac{l_B}{\sqrt{2}}\left(a + a^{+}\right) = \underline{\hat{X}(0)}$$

$$\hat{Y}(t) = -i\frac{l_B}{\sqrt{2}}\left(\hat{a}(t) - a^{+}(t)\right) = \hat{Y}(0) + \frac{l_B^2}{\hbar} eEt$$

$$= \underline{\hat{Y}(0) + \frac{E}{B}ct}$$

Drift in the y-direction with constant velocity $\underline{v_{drift} = \frac{E}{B}c}$

FYS 4110, 2006

Midterm exam, solutions

Problem 1, Spin coherent states

a) Eigenvalue equation $\hat{J}_- |\psi\rangle = \lambda |\psi\rangle$

$$|\psi\rangle = \sum_m c_m |j,m\rangle \qquad \text{expansion in } |j,m\rangle \text{ basis}$$

$m \leq j \Rightarrow$ there is a maximum value m_{max} in the expansion.

When $\hat{J}_-$ is applied to $|\psi\rangle$ that will reduce the max. value, $m_{max} \to m_{max} - 1$, since $\hat{J}_-$ lowers the m value

This creates a conflict between the RHS and LHS of the eigenvalue equation. Only solution: $\lambda = 0$

$$\Rightarrow |\psi\rangle = |j,-j\rangle$$

b)

$$(\Delta \vec{J})^2 = j(j+1)\hbar^2 - \langle \hat{\vec{J}} \rangle^2$$

min. value of $(\Delta \vec{J})^2 \Rightarrow$ max value of $\langle \hat{\vec{J}} \rangle^2$

Assume $\langle \hat{J}_x \rangle = \langle \hat{J}_y \rangle = 0$, $\langle \hat{J}_z \rangle = J$

We have $-j\hbar \leq \langle \hat{J}_z \rangle \leq j\hbar$

Equality: $\hat{J}_z |j,j\rangle = j\hbar |j,j\rangle$; $\hat{J}_z |j,-j\rangle = -j\hbar |j,-j\rangle$

Max. value of $\langle \hat{J}_z \rangle^2$; $j^2\hbar^2$ for $|j,j\rangle$ and $|j,-j\rangle$

Min value for $(\Delta \vec{J})^2$: $j(j+1)\hbar^2 - j^2\hbar^2 = j\hbar^2$

for $|j,j\rangle$ and $|j,-j\rangle$

c) The solution in b) is valid for any choice of the z-direction (rotational invariance).

Assume $\vec{n}$ is a unit vector in an arbitrary direction. We choose this to be the (new) z-axis: $\vec{n} = \vec{k}$

The results of b) applies to this situation and we translate to $\vec{n}$-variable:

$$\hat{J}_z = \vec{k} \cdot \hat{\vec{J}} = \vec{n} \cdot \hat{\vec{J}}$$

minimum uncertainty state:

$$\hat{J}_z \,|j,j\rangle = j\hbar\, |j,j\rangle$$

$$\Leftrightarrow \quad \vec{n}\cdot\hat{\vec{J}}\, |\vec{n},j\rangle = j\hbar\, |\vec{n},j\rangle$$

with $|\vec{n},j\rangle$ as max spin state in the $\vec{n}$-direction

$$\langle j,j \,|\, \hat{\vec{J}} \,|\, j,j\rangle = J\vec{k} \quad , \quad J = j\hbar$$

$$\Leftrightarrow \quad \langle \vec{n}\, j \,|\, \hat{\vec{J}} \,|\, \vec{n}\, j\rangle = J\vec{n}$$

d) Choose an arbitrary spin ½ state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad |\alpha|^2 + |\beta|^2 = 1$$

($|0\rangle$ = spin down, $|1\rangle$ = spin up in the z-direction

$$\langle \hat{\vec{J}} \rangle = \frac{\hbar}{2} (\beta^* \alpha^*)\, \vec{\sigma} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$= \frac{\hbar}{2} \Big\{ (\beta^* \alpha^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \vec{i} + (\beta^* \alpha^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \vec{j}$$

$$+ (\beta^* \alpha^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \vec{k} \Big\}$$

$$= \frac{\hbar}{2} \Big\{ (\alpha^*\beta + \alpha\beta^*)\, \vec{i} + i(\alpha^*\beta - \alpha\beta^*)\, \vec{j} + (|\beta|^2 - |\alpha|^2)\, \vec{k} \Big\}$$

$$\langle \vec{J} \rangle^2 = \frac{\hbar^2}{4}\left((\alpha^*\beta + \alpha\beta^*)^2 - (\alpha^*\beta - \alpha\beta^*)^2 + (|\beta|^2 - |\alpha|^2)^2 \right)$$

$$= \frac{\hbar^2}{4}\left(4|\alpha|^2|\beta|^2 + (|\beta|^2 - |\alpha|^2)^2 \right)$$

$$= \frac{\hbar^2}{4}\left(|\alpha|^2 + |\beta|^2 \right)^2$$

$$= \frac{\hbar^2}{4}$$

$$\Rightarrow (\Delta\vec{J})^2 = \frac{1}{2}\frac{3}{2}\hbar^2 - \frac{1}{4}\hbar^2 = \underline{\frac{1}{2}\hbar^2}$$

Result the same for all states (independent of α and β)

$\Rightarrow$ all states have min value (and max value) for $(\Delta\vec{J})^2$

e) Coherent state defined by

$$\vec{\sigma}\cdot\vec{n}\,|z\rangle = |z\rangle$$

with $|z\rangle = \alpha|0\rangle + \beta|1\rangle$ write this as a two-komponent eq. in the k-basis

$$\begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \qquad |\alpha|^2 + |\beta|^2 = 1$$

$$\Rightarrow \cos\theta\,\beta + e^{-i\varphi}\sin\theta\,\alpha = \beta$$

$$\Rightarrow \frac{\beta}{\alpha}(1-\cos\theta) = e^{-i\varphi}\sin\theta \qquad 1-\cos\theta = 2\sin^2\frac{\theta}{2}, \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\Rightarrow \frac{\beta}{\alpha}\sin\frac{\theta}{2} = e^{-i\varphi}\cos\theta/2$$

$$\frac{\beta}{\alpha} = e^{-i\varphi}\cot\theta/2 = z$$

$$\Rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = N\begin{pmatrix} z \\ 1 \end{pmatrix} \qquad N^2(|z|^2+1) = 1$$

$$= \frac{1}{\sqrt{|z|^2+1}}$$

$$\underline{\langle 0|z\rangle = \alpha = \frac{1}{\sqrt{1+|z|^2}}}, \qquad \underline{\langle 1|z\rangle = \beta = \frac{z}{\sqrt{1+|z|^2}}}$$

f) $\langle z|z_0\rangle = \sum_{k=0}^{1} \langle z|k\rangle\langle k|z_0\rangle$

$$= \frac{1+z^* z_0}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$$

$$\Rightarrow \underline{|\langle z|z_0\rangle|^2 = \frac{1+z^*z_0+zz_0^*+|z|^2|z_0|^2}{(1+|z|^2)(1+|z_0|^2)}}$$

g) $\int d^2z \frac{1}{(1+|z|^2)^2} |z\rangle\langle z|$

$$= \sum_{kk'} \int d^2z \frac{\langle k|z\rangle\langle z|k'\rangle}{(1+|z|^2)^3} |k\rangle\langle k'|$$

$$= \sum_{kk'} |k\rangle\langle k'| \int d^2z \frac{z^k z^{*k'}}{(1+|z|^2)^3}$$

change to polar coordinates: $z = e^{i\varphi} r$, $d^2z = d\varphi\, dr\, r$

$$= \sum_{kk'} |k\rangle\langle k'| \underbrace{\int_0^{2\pi} d\varphi\, e^{i\varphi(k-k')}}_{2\pi\delta_{kk'}} \int_0^\infty dr \frac{r^{(k+k'+1)}}{(1+r^2)^3}$$

change of variable $t = r^2 \Rightarrow r\,dr = \frac{1}{2}dt$

$$= \sum_k \pi \int_0^\infty dt \frac{t^k}{(1+t)^3} |k\rangle\langle k|$$

$k=0 \quad \int_0^\infty dt \frac{1}{(1+t)^3} = \left[-\frac{1}{2}\frac{1}{(1+t)^2}\right]_0^\infty = \frac{1}{2}$

$k=1 \quad \int_0^\infty dt \frac{t}{(1+t)^3} = \int_0^\infty dt \left(\frac{1}{(1+t)^2} - \frac{1}{(1+t)^3}\right) = 1-\frac{1}{2} = \frac{1}{2}$

$$\Rightarrow \int d^2z \frac{1}{(1+|z|^2)^2} |z\rangle\langle z| = \sum_k \frac{\pi}{2} |k\rangle\langle k| = \frac{\pi}{2}\mathbb{1}$$

$$\Rightarrow \underline{\int \frac{d^2z}{2\pi} \frac{4}{(1+|z|^2)^2} |z\rangle\langle z| = \mathbb{1}}$$

Problem 2, Entanglement in a three-particle system

a) Correlated state is not a product state:

$$\rho \neq \rho_A \otimes \rho_B \otimes \rho_C \qquad \text{mixed state}$$

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle \qquad \text{pure state}$$

$$\Rightarrow \langle \hat{A}\hat{B}\hat{C} \rangle \neq \langle \hat{A} \rangle \langle \hat{B} \rangle \langle \hat{C} \rangle$$

with $\hat{A}$ operating on subsystem A etc

Entangled state: not a statistical mixture of product states

$$\rho \neq \sum_k p_k \, \rho_k^A \otimes \rho_k^B \otimes \rho_k^C \qquad p_k \geq 0 \quad \sum_k p_k = 1$$

b)

$$\rho_A = \mathrm{Tr}_{BC}\, \rho = \tfrac{1}{2} \left(|u\rangle_A\langle u|_A + |d\rangle_A\langle d|_A \right) = \tfrac{1}{2} \mathbb{1}_A$$

$$\rho_{AB} = \mathrm{Tr}_A\, \rho = \tfrac{1}{2} \left(|uu\rangle_{BC}\langle uu|_{BC} + |dd\rangle_{BC}\langle dd|_{BC} \right)$$

Von Neuman entropy $S_A = S_{BC} = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = \underline{\log 2}$

Degree of entanglement measured by entropy of subsystem (when the total state is pure).

Subsystem A is maximally mixed $\Rightarrow S_A$ is maximal

$\Rightarrow$ A+BC is maximally entangled

Subsystem BC is a statistical mixture of product state

$\Rightarrow$ No separate entanglement between B and C

c)

$$|u\rangle = \tfrac{1}{\sqrt{2}} \left(|f\rangle + |b\rangle \right) = \tfrac{1}{\sqrt{2}} \left(|r\rangle + |\ell\rangle \right)$$

$$|d\rangle = \tfrac{1}{\sqrt{2}} \left(|f\rangle - |b\rangle \right) = -\tfrac{i}{\sqrt{2}} \left(|r\rangle - |\ell\rangle \right)$$

GHZ state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|uuu\rangle - |ddd\rangle\right)$$

$$= \frac{1}{2}\left(|ffb\rangle + |fbf\rangle + |bff\rangle + |bbb\rangle\right)$$

$$= \frac{1}{2}\left(|rrf\rangle + |llf\rangle + |rlb\rangle + |lrb\rangle\right)$$

d) In all three cases, the expressions for $|\psi\rangle$ show that if the spin component of B and C are determined (by measurement) then the spin component of A is also uniquely determined.

1) Measurement of the spin in the z-direction of <u>either</u> B or C will determine the spin in the z-direction for the two other particles. (Strict correlation in the z-component of the spin.)

2) Measurement of the spin in the x-direction for B <u>and</u> C will determine the spin in the x-direction for A (For example fb measured for BC implies f for A)

3) Measurement of the spin in the y-direction for B and the x-component for C will determine the y component for A. (For example lf measured for BC implies l for A)

e) $\sigma_x |u\rangle = |d\rangle$, $\sigma_x |d\rangle = |u\rangle$

$\sigma_y |u\rangle = i|d\rangle$, $\sigma_y |d\rangle = -i|u\rangle$

$\Rightarrow \hat{O}_1 |\psi\rangle = \hat{O}_2 |\psi\rangle = \hat{O}_3 |\psi\rangle = |\psi\rangle$

all three have eigenvalue 1

$$\hat{O}_1\hat{O}_2\hat{O}_3 = \sigma_x\sigma_y^2 \otimes \sigma_y\sigma_x\sigma_y \otimes \sigma_y^2\sigma_x$$

$$\sigma_y^2 = \mathbb{1}, \quad \sigma_x\sigma_y = -\sigma_y\sigma_x$$

$$\Rightarrow \quad \hat{O}_1\hat{O}_2\hat{O}_3 = -\sigma_x \otimes \sigma_x \otimes \sigma_x = -\hat{O}_4 \qquad \text{eigenvalue of } \underline{\hat{O}_4 = -1}$$

f) Eigenvalue equations for $\hat{O}_1, \hat{O}_2, \hat{O}_3$

1: $m_x^A m_y^B m_y^C = 1$

2: $m_y^A m_x^B m_y^C = 1$

3: $m_y^A m_y^B m_x^C = 1$

product of ~~equat~~ equations

$$m_x^A m_y^{A\,2} m_x^B m_y^{B\,2} m_x^C m_y^{C\,2} = 1$$

$$m_y^{A\,2} = m_y^{B\,2} = m_y^{C\,2} = 1 \Rightarrow$$

$$\underline{m_x^A m_x^B m_x^C = 1}$$

Eigenvalue equation for $\hat{O}_4$

$$\underline{m_x^A m_x^B m_x^C = -1}$$

contradicts equations for $\hat{O}_1, \hat{O}_2, \hat{O}_3$

Cannot assume spin components to have sharp, but undetermined values before the measurements.

FYS 4110 Midterm Exam 2007

Solutions

Problem 1, Density operators

a) Density operator, matrix form

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

$$\rho_{11} = \langle + | \rho | + \rangle = \underline{\tfrac{1}{2}(1+z)}$$

$$\rho_{12} = \langle + | \rho | - \rangle = \underline{\tfrac{1}{2}(x-iy)}$$

$$\rho_{21} = \langle - | \rho | + \rangle = \underline{\tfrac{1}{2}(x+iy)}$$

$$\rho_{22} = \langle - | \rho | - \rangle = \underline{\tfrac{1}{2}(1-z)}$$

b) Reduced density matrices

$$\rho^A = \mathrm{Tr}_B\,\rho = \frac{1}{4}\Big(\mathbb{1}\cdot \mathrm{Tr}_B \mathbb{1} + \sum_i a_i \sigma_i \,\mathrm{Tr}_B \mathbb{1} + \sum_j b_j \mathbb{1}\,\mathrm{Tr}_B \sigma_j + \sum_{ij} c_{ij}\sigma_i\,\mathrm{Tr}_B\sigma_j\Big)$$

use: $\mathrm{Tr}\,\mathbb{1} = 2$ (2x2 matrix)

$\mathrm{Tr}\,\sigma_i = 0 \quad i = 1, 2, 3$

$$\Rightarrow \quad \rho^A = \underline{\tfrac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma})}$$

In the same way

$$\rho^B = \underline{\tfrac{1}{2}(\mathbb{1} + \vec{b}\cdot\vec{\sigma})}$$

Completely uncorrelated means ρ is a product,

$$\rho = \rho^A \otimes \rho^B$$

$$= \frac{1}{4}\Big(\mathbb{1}\otimes\mathbb{1} + \sum_i a_i\sigma_i\otimes\mathbb{1} + \sum_j b_j \mathbb{1}\otimes\sigma_j + \sum_{ij} a_i b_j\,\sigma_i\otimes\sigma_j\Big)$$

The two subsystems are uncorrelated if $\underline{c_{ij} = a_i b_j}$

c) Density operators for the Bell states

$$\rho_{c\pm} = |c\pm\rangle\langle c\pm| = \tfrac{1}{2}\left(|++\rangle\langle ++| \pm |++\rangle\langle --| \pm |--\rangle\langle ++| + |--\rangle\langle --|\right)$$

$$\rho_{a\pm} = |a\pm\rangle\langle a\pm| = \tfrac{1}{2}\left(|+-\rangle\langle +-| \pm |+-\rangle\langle -+| \pm |-+\rangle\langle +-| + |-+\rangle\langle -+|\right)$$

Expressed in terms of Pauli matrices

$$|+\rangle\langle +| = \tfrac{1}{2}(\mathbb{1}+\sigma_z) \qquad |-\rangle\langle -| = \tfrac{1}{2}(\mathbb{1}-\sigma_z)$$

$$|+\rangle\langle -| = \tfrac{1}{2}(\sigma_x+i\sigma_y) \qquad |-\rangle\langle +| = \tfrac{1}{2}(\sigma_x-i\sigma_y)$$

for composite system

$$|++\rangle\langle ++| = |+\rangle\langle +| \otimes |+\rangle\langle +| = \tfrac{1}{4}(\mathbb{1}+\sigma_z)\otimes(\mathbb{1}+\sigma_z)$$

$$= \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\sigma_z)$$

$$|--\rangle\langle --| = \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\sigma_z)$$

$$\Rightarrow \quad |++\rangle\langle ++| + |--\rangle\langle --| = \tfrac{1}{2}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$$

$$|++\rangle\langle --| = |+\rangle\langle -| \otimes |+\rangle\langle -| = \tfrac{1}{4}(\sigma_x+i\sigma_y)\otimes(\sigma_x+i\sigma_y)$$

$$= \tfrac{1}{4}(\sigma_x\otimes\sigma_x + i\sigma_x\otimes\sigma_y + i\sigma_y\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$|--\rangle\langle ++| = \tfrac{1}{4}(\sigma_x\otimes\sigma_x - i\sigma_x\otimes\sigma_y - i\sigma_y\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \quad |++\rangle\langle --| + |--\rangle\langle ++| = \tfrac{1}{2}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \quad \rho_{c\pm} = \underline{\tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} \pm \sigma_x\otimes\sigma_x \mp \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)}$$

$$|+-\rangle\langle +-| = |+\rangle\langle +| \otimes |-\rangle\langle -| = \tfrac{1}{4}(\mathbb{1}+\sigma_z)\otimes(\mathbb{1}-\sigma_z)$$

$$= \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$

$$|-+\rangle\langle -+| = \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z)$$

$$\Rightarrow \quad |+-\rangle\langle +-| + |-+\rangle\langle -+| = \tfrac{1}{2}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\sigma_z)$$

$$|+-\rangle\langle-+| = |+-\rangle \otimes \langle -+| = \tfrac{1}{4}(\sigma_x + i\sigma_y)\otimes(\sigma_x - i\sigma_y)$$

$$= \tfrac{1}{4}(\sigma_x\otimes\sigma_x - i\sigma_x\otimes\sigma_y + i\sigma_y\otimes\sigma_x + \sigma_y\otimes\sigma_y)$$

$$|-+\rangle\langle+-| = \tfrac{1}{4}(\sigma_x\otimes\sigma_x + i\sigma_x\otimes\sigma_y - i\sigma_y\otimes\sigma_y + \sigma_y\otimes\sigma_y)$$

$$\Rightarrow |+-\rangle\langle-+| + |-+\rangle\langle+-| = \tfrac{1}{2}(\sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \underline{\rho_{a\pm} = \tfrac{1}{4}(\mathbb{1}\otimes\mathbb{1} \pm \sigma_x\otimes\sigma_x \pm \sigma_y\otimes\sigma_y \mp \sigma_z\otimes\sigma_z)}$$

Note: no terms of the form $\sigma_i\otimes\mathbb{1}$ or $\mathbb{1}\otimes\sigma_i$ $(\vec{a}=\vec{b}=0)$

$\Rightarrow$ $\underline{\rho^A = \frac{1}{2}\mathbb{1}}$, $\underline{\rho^B = \frac{1}{2}\mathbb{1}}$ $\Rightarrow$ entropy $\underline{S^A = S^B = \ln 2}$

for all the four Bell states

The Bell states are pure states: Correlations are due to entanglement; $S=0$ (entropy of full system)

$S^A = S^B$ are maximal for subsystems $\Rightarrow$ maximal entanglement.

d) Check of conditions:

1) $\rho = \rho^+$, 2) $\rho \geq 0$ (non-neg. eigenv.), 3) $\mathrm{Tr}\rho = 1$

Satisfied for ρ_1 and ρ_2

1) $\rho^+ = x^*\rho_1^+ + (1-x^*)\rho_2^+ = x\rho_1 + (1-x)\rho_2 = \rho$ (x real)

2) $0 < x < 1 \Rightarrow x > 0$ & $1-x > 0$

positive combination of positive operators

$\Rightarrow$ general state $\langle\psi|\rho|\psi\rangle = x\langle\psi|\rho_1|\psi\rangle + (1-x)\langle\psi|\rho_2|\psi\rangle \geq 0$

$\Rightarrow \rho \geq 0$

3) $\mathrm{Tr}\rho = x\,\mathrm{Tr}\rho_1 + (1-x)\,\mathrm{Tr}\rho_2 = x + (1-x) = \underline{1}$

If $x<0$ or $1-x>0$: 1) and 3) still ok, but positivity not satisfied.

e) Choose f. ex. ρ_{c+} and ρ_{c-} :

$$\rho = \frac{1}{2}(\rho_{c+} + \rho_{c-})$$
$$= \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$$
$$= \frac{1}{8}\left((\mathbb{1}+\sigma_z)\otimes(\mathbb{1}+\sigma_z) + (\mathbb{1}-\sigma_z)\otimes(1-\sigma_z)\right)$$

This is of the form

similar results for other choices.

$$\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B$$

seperable, per definition non-entangled.

f) The Bell states have density operators that are all combinations of $\sigma_x\otimes\sigma_x$, $\sigma_y\otimes\sigma_y$, $\sigma_z\otimes\sigma_z$ (and identity $\mathbb{1}$). These all commute:

$$\sigma_x\sigma_y = i\sigma_z = -\sigma_y\sigma_x \Rightarrow$$
$$(\sigma_x\otimes\sigma_x)(\sigma_y\otimes\sigma_y) = \sigma_x\sigma_y\otimes\sigma_x\sigma_y = -\sigma_z\otimes\sigma_z$$
$$(\sigma_y\otimes\sigma_y)(\sigma_x\otimes\sigma_x) = \sigma_y\sigma_x\otimes\sigma_y\sigma_x = -\sigma_z\otimes\sigma_z$$
$$\Rightarrow [\sigma_x\otimes\sigma_x, \sigma_y\otimes\sigma_y] = 0$$ similar argument for other operators

More general argument:

Orthogonal pure states $\rho_1 = |\psi_1\rangle\langle\psi_1|$, $\rho_2 = |\psi_2\rangle\langle\psi_2|$

$$\Rightarrow \rho_1\rho_2 = |\psi_1\rangle\underbrace{\langle\psi_1|\psi_2\rangle}_{=0}\langle\psi_2| = 0 = \rho_2\rho_1$$

All four Bell states are orthogonal $\Rightarrow$ density op. commute

$\Rightarrow$ all linear combinations of these commute.

Problem 2, Jaynes-Cummings model

a) Eigenstates of H_0

$$H_0|m,n\rangle = \hbar\left(\tfrac{1}{2}m\omega_0 + n\omega\right)|m,n\rangle \quad m=\pm 1,\ n=0,1,2,\dots$$

$$|1\rangle = |1,n-1\rangle, \quad |2\rangle = |-1,n\rangle \Rightarrow$$

$$H_0|1\rangle = \hbar\left(\tfrac{1}{2}\omega_0 + (n-1)\omega\right)|1\rangle \equiv \left(\tfrac{1}{2}\hbar\Delta + \varepsilon\right)|1\rangle$$

$$H_0|2\rangle = \hbar\left(-\tfrac{1}{2}\omega_0 + n\omega\right)|2\rangle \equiv \left(-\tfrac{1}{2}\hbar\Delta + \varepsilon\right)|2\rangle$$

$$\Rightarrow \underline{\Delta = \omega_0 - \omega} \quad \underline{\varepsilon = (n-\tfrac{1}{2})\hbar\omega}$$

$$H_1|1\rangle = i\hbar\lambda\,\sigma_-|1\rangle \otimes a^\dagger|n-1\rangle$$

$$= i\hbar\lambda\sqrt{n}\,|2\rangle \qquad = \tfrac{1}{2}i\hbar g|2\rangle$$

$$H_1|2\rangle = -i\hbar\lambda\sqrt{n}\,|1\rangle \qquad \Big\} \Rightarrow \underline{g = 2\lambda\sqrt{n}}$$

2x2 matrix form

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \underline{\frac{1}{2}\hbar\begin{pmatrix} \Delta & -ig \\ ig & -\Delta \end{pmatrix} + \varepsilon\mathbb{1}}$$

b) Eigenvalue problem

new parameters $\Delta = \Omega\cos\theta$, $g = \Omega\sin\theta \Rightarrow \Omega = \sqrt{\Delta^2+g^2}$

$$\Rightarrow H = \frac{1}{2}\hbar\Omega\begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix} + \varepsilon\mathbb{1}$$

eigenvalue problem to solve:

$$\begin{pmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \Lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} \cos\theta - \Lambda & -i\sin\theta \\ i\sin\theta & -\cos\theta - \Lambda \end{vmatrix} = 0$$

$$(\cos\theta-\Lambda)(-\cos\theta-\Lambda)-\sin^2\theta=0$$

$$\Lambda^2-\cos^2\theta-\sin^2\theta=0$$

$$\Lambda^2=1 \Rightarrow \Lambda_\pm=\pm 1$$ two eigenvalues

Energies

$$E_\pm=\tfrac{1}{2}\hbar\Omega\Lambda_\pm+\varepsilon$$

$$=(n-\tfrac{1}{2})\hbar\omega\pm\tfrac{1}{2}\hbar\sqrt{(\omega_0-\omega)^2+4n\lambda^2}$$

Eigenvectors

$$\cos\theta\, a_\pm - i\sin\theta\, b_\pm = \pm a_\pm$$

$$\mp(1\mp\cos\theta)\,a_\pm = i\sin\theta\, b_\pm$$

Use :

$$\sin\theta = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$1-\cos\theta = 2\sin^2\tfrac{\theta}{2}$$

$$1+\cos\theta = 2\cos^2\tfrac{\theta}{2}$$

$$\Rightarrow 2\sin^2\tfrac{\theta}{2}\,a_+ = -i\,2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\,b_+$$

$$\sin\tfrac{\theta}{2}\,a_+ = -i\cos\tfrac{\theta}{2}\,b_+$$

matrix $\underline{\psi_+=\begin{pmatrix}a_+\\b_+\end{pmatrix}=\begin{pmatrix}i\cos\frac{\theta}{2}\\-\sin\frac{\theta}{2}\end{pmatrix}}$

$$2\cos^2\tfrac{\theta}{2}\,a_- = i\,2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\,b_-$$

$$\cos\tfrac{\theta}{2}\,a_- = i\sin\tfrac{\theta}{2}\,b_-$$

matrix $\underline{\psi_-=\begin{pmatrix}a_-\\b_-\end{pmatrix}=\begin{pmatrix}i\sin\frac{\theta}{2}\\\cos\frac{\theta}{2}\end{pmatrix}}$

General state

$$\psi = d_+ \psi_+ + d_- \psi_-$$

$$= \begin{pmatrix} i(d_+ \cos\frac{\theta}{2} + d_- \sin\frac{\theta}{2}) \\ -d_+ \sin\frac{\theta}{2} + d_- \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Initial condition

$$c_1(0) = 0 \qquad c_2(0) = 1$$

$$\Rightarrow \quad d_+(0) \cos\frac{\theta}{2} + d_-(0) \sin\frac{\theta}{2} = 0$$

$$-d_+(0) \sin\frac{\theta}{2} + d_-(0) \cos\frac{\theta}{2} = 1$$

$$\Rightarrow \quad d_+(0) = -\sin\frac{\theta}{2}, \quad d_-(0) = \cos\frac{\theta}{2}$$

Time evolution

$$d_+(t) = e^{-\frac{i}{\hbar}E_+ t} d_+(0) = -\sin\frac{\theta}{2}\, e^{-\frac{i}{2}\Omega t}\, e^{-\frac{i}{\hbar}\varepsilon t}$$

$$d_-(t) = e^{-\frac{i}{\hbar}E_- t} d_-(0) = \cos\frac{\theta}{2}\, e^{\frac{i}{2}\Omega t}\, e^{-\frac{i}{\hbar}\varepsilon t}$$

$$\Rightarrow \quad c_1(t) = i\left(d_+(t) \cos\frac{\theta}{2} + d_-(t) \sin\frac{\theta}{2}\right)$$

$$= i e^{-\frac{i}{\hbar}\varepsilon t}\left(-\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-\frac{i}{2}\Omega t} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\, e^{\frac{i}{2}\Omega t}\right)$$

$$= \underline{-e^{-\frac{i}{\hbar}\varepsilon t} \sin\theta \sin\frac{\Omega}{2}t}$$

$$c_2(t) = -d_+(t) \sin\frac{\theta}{2} + d_-(t) \cos\frac{\theta}{2}$$

$$= e^{-\frac{i}{\hbar}\varepsilon t}\left(\sin^2\frac{\theta}{2}\, e^{-\frac{i}{2}\Omega t} + \cos^2\frac{\theta}{2}\, e^{\frac{i}{2}\Omega t}\right)$$

$$= \underline{e^{-\frac{i}{2}\varepsilon t}\left(\cos\frac{\Omega}{2}t + i\cos\theta \sin\frac{\Omega}{2}t\right)}$$

$$\underline{|c_1(t)|^2 = \sin^2\theta \sin^2\frac{\Omega}{2}t}$$

d) The atom is initially in the lowest energy state. The interaction with the electromagnetic field introduces oscillations between this state and the excited atomic state, with oscillation frequency Ω.

The situation is similar to that of Sect. 1.3.2 of the lecture notes, where the oscillations are induced by a time-dependent magnetic field.

Connection between the two expressions

$$g = \omega_1 \Rightarrow$$

$$2\lambda\sqrt{n} = -\frac{eB_1}{m_e c} \Rightarrow \underline{B_1 = \text{const}\cdot\sqrt{n}}$$

the amplitude of the magnetic field is proportional to the square root of the photon number.

e) $$|\psi(t)\rangle = c_1(t)\,|+1\rangle_A \otimes |n-1\rangle_B + c_2(t)\,|-1\rangle_A \otimes |n\rangle_B$$

$$\Rightarrow \rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$= |c_1(t)|^2\,|+1\rangle_A\langle+1|_A \otimes |n-1\rangle_B\langle n-1|_B$$

$$+ |c_2(t)|^2\,|-1\rangle_A\langle-1|_A \otimes |n\rangle_B\langle n_B|$$

$$+ c_1(t)\,c_2(t)^*\,|+1\rangle_A\langle-1|_A \otimes |n-1\rangle_B\langle n|_B$$

$$+ c_1^*(t)\,c_2(t)\,|-1\rangle_A\langle+1|_A \otimes |n\rangle_B\langle n-1|_B$$

f) Reduced density matrix

$$\rho_A = \mathrm{Tr}_B\,\rho = {}_B\langle n-1|\,\rho\,|n-1\rangle_B + {}_B\langle n|\,\rho\,|n\rangle_B$$

$$= |c_1|^2\,|+1\rangle_A\langle +1|_A + |c_2|^2\,|-1\rangle_A\langle -1|_A$$

matrix form

$$\rho_A(t) = \begin{pmatrix} |c_1(t)|^2 & 0 \\ 0 & |c_2(t)|^2 \end{pmatrix}$$

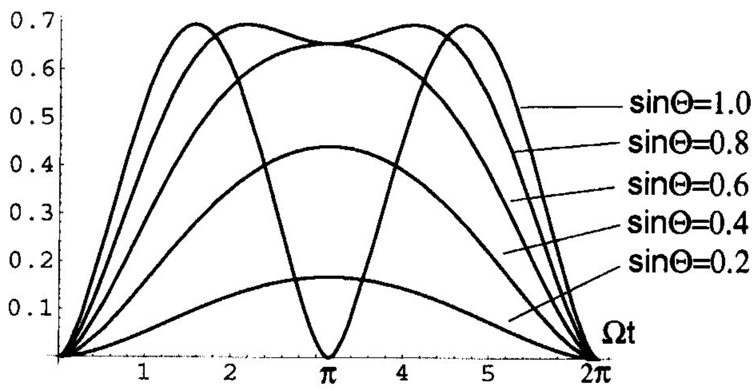
Entropy

$$S_A(t) = -\left(|c_1(t)|^2 \log|c_1(t)|^2 + |c_2(t)|^2 \log|c_2(t)|^2\right)$$

$$= \underline{-\left(|c_1(t)|^2 \log|c_1(t)|^2 + (1-|c_1(t)|^2)\log(1-|c_1(t)|^2)\right)}$$

$$|c_1(t)|^2 = \sin^2\theta\,\sin^2\tfrac{\Omega}{2}t$$

Plot of $S_A(t)$ for different values of $\sin\theta$:



For a pure state (of the composite system), the entropy of the subsystem gives a measure of the degree of entanglement. The figure shows: Entanglement increases from a minimum at $\Omega t = 0$ $(2\pi, 4\pi, \ldots)$. A maximum is reached at $\Omega t = \pi$ for small θ ($\sin\theta < \frac{1}{\sqrt{2}}$). For larger θ $\Omega t = \pi$ is instead a minimum and the maxima moves towards $\Omega t = \frac{\pi}{2}, 3\frac{\pi}{2}$.

FYS4110 Midterm Exam 2008

Solutions

Problem 1 Spin splitting in positronium

a) $\langle ij | \vec{\Sigma}_e \cdot \vec{\Sigma}_p | k\ell \rangle$

$$= \sum_{mn} \langle ij | \vec{\sigma}_e \otimes \mathbb{1}_p | mn \rangle \langle mn | \mathbb{1}_e \otimes \vec{\sigma}_p | k\ell \rangle$$

$$= \sum_{mn} \left(\langle i | \vec{\sigma}_e | m \rangle \delta_{jn} \right) \cdot \left(\delta_{mk} \langle n | \vec{\sigma}_p | \ell \rangle \right)$$

$$= \underline{\langle i | \vec{\sigma}_e | k \rangle \cdot \langle j | \vec{\sigma}_p | \ell \rangle}$$

b) matrix elements

$\vec{\sigma} = \sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k} \quad \Rightarrow$

$\langle + | \vec{\sigma} | + \rangle = \vec{k} \;, \quad \langle - | \vec{\sigma} | - \rangle = -\vec{k}$

$\langle + | \vec{\sigma} | - \rangle = \vec{i} - i\vec{j} \;, \quad \langle - | \vec{\sigma} | + \rangle = \vec{i} + i\vec{j}$

$\Rightarrow \quad \langle ++ | \vec{\Sigma}_e \cdot \vec{\Sigma}_p | ++ \rangle = \vec{k} \cdot \vec{k} = 1$

$\langle ++ | \text{ -"- } | +- \rangle = \vec{k} \cdot (\vec{i} - i\vec{j}) = 0$

$\langle ++ | \text{ -"- } | -+ \rangle = \text{ -"- } = 0$

$\langle ++ | \text{ -"- } | -- \rangle = (\vec{i} - i\vec{j}) \cdot (\vec{i} - i\vec{j}) = 0$

$\langle +- | \text{ -"- } | +- \rangle = \vec{k} \cdot (-\vec{k}) = -1$

$\langle +- | \text{ -"- } | -+ \rangle = (\vec{i} - i\vec{j}) \cdot (\vec{i} + i\vec{j}) = 2$

$\langle +- | \text{ -"- } | -- \rangle = (\vec{i} - i\vec{j}) \cdot (-\vec{k}) = 0$

$\langle -+ | \text{ -"- } | -+ \rangle = (-\vec{k}) \cdot \vec{k} = -1$

$\langle -+ | \text{ -"- } | -- \rangle = (-\vec{k}) \cdot (\vec{i} - i\vec{j}) = 0$

$\langle -- | \text{ -"- } | -- \rangle = (-\vec{k}) \cdot (-\vec{k}) = 1$

other matrix elements determined by hermiticity of $\vec{\Sigma}_e \cdot \vec{\Sigma}_p$

Matrix representation

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) From b) follows

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |0,0\rangle = \cancel{4} \frac{1}{\sqrt{2}} \left(\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |+-\rangle - \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |-+\rangle \right)$$

$$= \frac{\hbar^2}{4} \frac{1}{\sqrt{2}} \left((-|+-\rangle + 2|-+\rangle) - (-|-+\rangle + 2|+-\rangle) \right)$$

$$= -\frac{3}{4} \hbar^2 |0,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,1\rangle = \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |++\rangle = \frac{\hbar^2}{4} |1,1\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,0\rangle = \frac{\hbar^2}{4} \frac{1}{\sqrt{2}} \left((-|+-\rangle + 2|-+\rangle) + (-|-+\rangle + 2|+-\rangle) \right)$$

$$= \frac{1}{4} \hbar^2 |1,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,-1\rangle = \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |--\rangle = \frac{\hbar^2}{4} |1,-1\rangle$$

matrix form in the spin basis

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Total spin $\hat{\vec{S}} = \hat{\vec{S}}_e + \hat{\vec{S}}_p \Rightarrow \hat{\vec{S}}^2 = \hat{\vec{S}}_e^2 + \hat{\vec{S}}_p^2 + 2 \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$

$$\hat{\vec{S}}_e^2 = \frac{\hbar^2}{4} \vec{\sigma}_e^2 \otimes \mathbb{1}_p = 3 \frac{\hbar^2}{4} \mathbb{1}_e \otimes \mathbb{1}_p = \frac{3}{4} \hbar^2 \mathbb{1} = \hat{\vec{S}}_p^2$$

$$\Rightarrow \hat{\vec{S}}^2 = \frac{3}{2} \hbar^2 \mathbb{1} + 2 \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

$$\Rightarrow \hat{\vec{S}}^2 |0,0\rangle = 0 , \quad \hat{\vec{S}}^2 |1,m\rangle = 2\hbar^2 |1,m\rangle \qquad m = 0, \pm 1$$

$$\hat{S}_z |0,0\rangle = 0 , \quad \hat{S}_z |1,m\rangle = m\hbar |1,m\rangle$$

$\vec{S}^2 = s(s+1)\hbar^2 \Rightarrow s = 0$ for $|0,0\rangle$, $s = 1$ for $|1,m\rangle$

d) Need to find the matrix elements of $(S_e)_z - (S_p)_z \equiv D$

$$D|1,1\rangle = D|1,-1\rangle = 0$$

$$D|0,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle - (-2)|-+\rangle\right) = \hbar|1,0\rangle$$

$$D|1,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle + (-2)|-+\rangle\right) = \hbar|0,0\rangle$$

mixes only $|0,0\rangle$ and $|1,0\rangle$

Hamiltonian in the spin basis

$$H = \begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & 0 & \lambda\hbar^2 & 0 \\ 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 & 0 \\ \lambda\hbar^2 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 \\ 0 & 0 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}$$

e) · $|1,1\rangle$ and $|1,-1\rangle$ are eigenvectors with eigenvalues $\underline{E = E_0 + \frac{1}{4}\hbar^2\kappa}$ (indep. of λ)

Eigenvalue problem for the remaining two states

$$\begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & \lambda\hbar^2 \\ \lambda\hbar^2 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = E\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

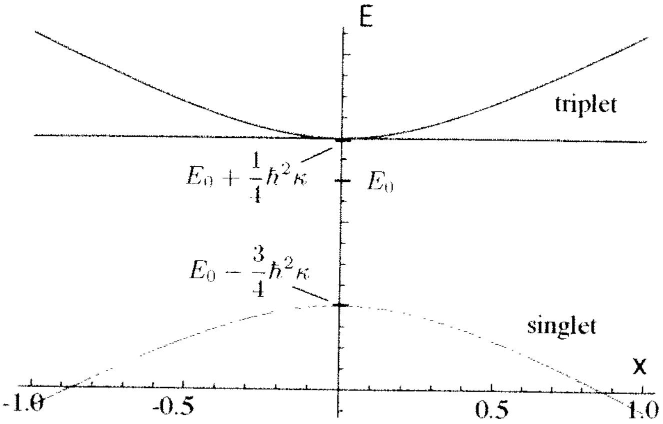
write this as $\left(E_0 - \frac{1}{4}\hbar^2\kappa\right)\mathbb{1} + \frac{1}{2}\hbar^2\kappa\begin{pmatrix} -1 & 2x \\ 2x & 1\end{pmatrix}$ $\qquad x = \lambda/\kappa$

$$\Rightarrow \begin{pmatrix} -1 & 2x \\ 2x & 1\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \mu\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \quad \text{with} \quad E = E_0 - \frac{1}{4}\hbar^2\kappa + \frac{1}{2}\hbar^2\kappa\mu$$

eigenvalues $\begin{vmatrix} -1-\mu & 2x \\ 2x & 1-\mu\end{vmatrix} = 0 \Rightarrow \mu^2 = 4x^2 + 1$

$$E_\pm = E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\kappa\sqrt{4x^2+1}$$

$$= \underline{E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\sqrt{\kappa^2 + 4\lambda^2}}$$



f) $$\hat{\rho}_A = |A\rangle\langle A| = |a|^2 |+-\rangle\langle +-| + |b|^2 |-+\rangle\langle -+| + ab^* |+-\rangle\langle -+| + a^* b |-+\rangle\langle +-|$$

$$\hat{\rho}_B = |B\rangle\langle B| = |b|^2 |+-\rangle\langle +-| + |a|^2 |-+\rangle\langle -+| - ab^* |+-\rangle\langle -+| - a^* b |-+\rangle\langle +-|$$

Reduced density operators

$$\hat{\rho}_{Ae} = \mathrm{Tr}_p\, \hat{\rho}_A = |a|^2 |+\rangle\langle +| + |b|^2 |-\rangle\langle -|$$

$$\hat{\rho}_{Ap} = \mathrm{Tr}_e\, \hat{\rho}_A = |a|^2 |-\rangle\langle -| + |b|^2 |+\rangle\langle +|$$

$$\hat{\rho}_{Be} = \mathrm{Tr}_p\, \hat{\rho}_B = |b|^2 |+\rangle\langle +| + |a|^2 |-\rangle\langle -|$$

$$\hat{\rho}_{Bp} = \mathrm{Tr}_e\, \hat{\rho}_B = |b|^2 |-\rangle\langle -| + |a|^2 |+\rangle\langle +|$$

g. Entropy

$$S_{Ae} = S_{Ap} = S_{Be} = S_{Bp} = -\left(|a|^2 \log |a|^2 + |b|^2 \log |b|^2\right)$$

$$= \underline{-\left(|a|^2 \log |a|^2 + (1-|a|^2) \log (1-|a|^2)\right)}$$

g) Eigenstates

$$|A\rangle = \alpha|0,0\rangle + \beta|1,0\rangle = a|+-\rangle + b|-+\rangle$$

$$\Rightarrow \quad a = \frac{\alpha+\beta}{\sqrt{2}} \quad , \quad b = \frac{\alpha-\beta}{\sqrt{2}}$$

α, β determined by eigenvalue eq. in e):

$$-\alpha + 2x\beta = \mu\alpha \quad \Rightarrow \quad \beta = \frac{\mu+1}{2x}\alpha$$

$\mu = \pm\sqrt{4x^2+1}$; choose $\mu = -\sqrt{4x^2+1}$ (+ gives $|B\rangle$)

gives $\beta \to 0$ for $x \to 0$

Note α, β real.

Normalization : $\alpha^2 + \beta^2 = \left(1 + \left(\frac{\mu+1}{2x}\right)^2\right)\alpha^2 = 1$

$$\Rightarrow \quad \alpha^2 = \frac{4x^2}{4x^2+(\mu+1)^2}$$

$$a^2 = \frac{1}{2}\left(1+\frac{\mu+1}{2x}\right)^2\alpha^2 = \frac{1}{2}\,\frac{(2x+\mu+1)^2}{4x^2+(\mu+1)^2}$$

$$(2x+\mu+1)^2 = 4x^2+1+4x+\mu^2+2(2x+1)\mu$$
$$= 2\left(\mu^2 + 2x(\mu+1)+\mu\right) = 2(\mu+1)(\mu+2x)$$

$$4x^2+(\mu+1)^2 = 4x^2+1+\mu^2+2\mu = 2(\mu^2+\mu) = 2\mu(\mu+1)$$

$$\Rightarrow \quad a^2 = \frac{1}{2}\,\frac{2(\mu+2x)(\mu+1)}{2\mu(\mu+1)} = \underline{\frac{1}{2}\left(1+\frac{2x}{\sqrt{4x^2+1}}\right)}$$

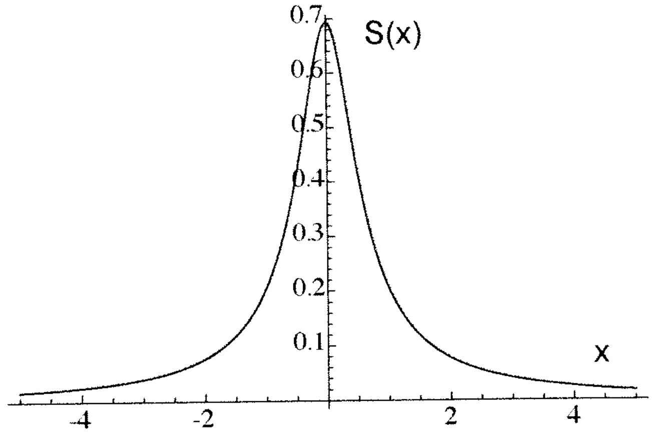
$$b^2 = 1-a^2 = \frac{1}{2}\left(1-\frac{2x}{\sqrt{4x^2+1}}\right)$$

Entropy of reduced density matrices

$$\underline{S(x) = -\left[a(x)^2\log a(x)^2 + b(x)^2\log b(x)^2\right]}$$

For $x = 0$: $\hat{\rho}_{Ae} = \hat{\rho}_{Be} = \frac{1}{2}\mathbb{1}_e$, $\hat{\rho}_{Ap} = \hat{\rho}_{Bp} = \frac{1}{2}\mathbb{1}_p$

maximal entanglement $S(0) = \log 2$



Entanglement of states |A> and |B> as functions of $x = \lambda/\kappa$

Problem 2 Driven harmonic oscillator

a) $$\hat{\mathcal{U}} e^{\hat{A}} \hat{\mathcal{U}}^{-1} = \hat{\mathcal{U}} (\mathbb{1} + \hat{A} + \tfrac{1}{2}\hat{A}^2 + \dots) \hat{\mathcal{U}}^{-1}$$
$$= \mathbb{1} + \hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^{-1} + \tfrac{1}{2}(\hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^{-1})^2 + \dots = e^{\hat{\mathcal{U}}\hat{A}\hat{\mathcal{U}}^{-1}}$$

special case: $\hat{\mathcal{U}}_0(t)\hat{D}(z)\hat{\mathcal{U}}_0(t)^+ \qquad \hat{\mathcal{U}}_0(t)^+ = \hat{\mathcal{U}}_0(t)^{-1}$

$$= \hat{\mathcal{U}}_0(t)\, e^{z\hat{a}^+ - z^*\hat{a}}\, \hat{\mathcal{U}}_0(t) = e^{\hat{\mathcal{U}}_0(t)(z\hat{a}^+ - z^*\hat{a})\hat{\mathcal{U}}_0(t)^+}$$

$$\hat{\mathcal{U}}_0(t)\hat{a}\hat{\mathcal{U}}_0(t)^+ = e^{-i\omega_0 t(\hat{a}^+\hat{a}+\frac{1}{2})}\, \hat{a}\, e^{i\omega_0 t(\hat{a}^+\hat{a}+\frac{1}{2})}$$
$$= \hat{a} - i\omega_0 t\,[\hat{a}^+\hat{a} + \tfrac{1}{2}, \hat{a}] + \tfrac{1}{2}(-i\omega_0 t)^2\,[\hat{a}^+\hat{a}+\tfrac{1}{2}, [\hat{a}^+\hat{a}+\tfrac{1}{2}, \hat{a}]] + \dots$$
$$= \hat{a} + i\omega_0 t\,\hat{a} + \tfrac{1}{2}(i\omega_0 t)^2\hat{a} + \dots$$
$$= e^{i\omega t}\hat{a}$$
$$\Rightarrow \hat{\mathcal{U}}_0(t)\hat{a}^+\hat{\mathcal{U}}_0(t)^+ = e^{-i\omega_0 t}\hat{a}^+$$
$$\Rightarrow \hat{\mathcal{U}}_0(t)\hat{D}(z)\hat{\mathcal{U}}_0(t)^+ = \hat{D}(z e^{-i\omega_0 t})$$

$$|\psi(t)\rangle = \hat{\mathcal{U}}_0(t)|z_0\rangle = \hat{\mathcal{U}}_0(t)\hat{D}(z_0)|0\rangle$$
$$= \hat{\mathcal{U}}_0(t)\hat{D}(z_0)\hat{\mathcal{U}}_0(t)^+\hat{\mathcal{U}}(t_0)|0\rangle$$
$$= \hat{D}(z_0 e^{-i\omega_0 t})\, e^{-\frac{i}{2}\omega_0 t}|0\rangle$$
$$= \underline{e^{-\frac{i}{2}\omega_0 t}|z_0 e^{-i\omega_0 t}\rangle}$$

remains a coherent state during the evolution.

b) $$\dot{\hat{x}} = \frac{i}{\hbar}[\hat{H}, \hat{x}] = \frac{i}{\hbar}\frac{1}{2m}[\hat{p}^2, \hat{x}] = \frac{i}{\hbar}\frac{1}{2m}(\hat{p}[\hat{p},\hat{x}] + [\hat{p},\hat{x}]\hat{p})$$
$$= \frac{\hat{p}}{m}$$

$$\dot{\hat{p}} = \frac{i}{\hbar}[\hat{H}, \hat{p}] = -\frac{d}{dx}\left(\tfrac{1}{2}m\omega_0^2\hat{x}^2 + W(\hat{x},t)\right) = -m\omega_0^2\hat{x} - \frac{\partial W}{\partial x}(\hat{x},t)$$

$$\Rightarrow \underline{m\ddot{\hat{x}} + m\omega_0^2\hat{x} = -\frac{\partial \hat{W}}{\partial x} = -A\, \overset{\sin}{\cancel{\cos}}\,\omega t\, \mathbb{1}}$$

Driven harmonic oscillator, force $f(t) = -A\, \overset{\sin}{\cancel{\cos}}\,\omega t$

c) Time evolution in the Schrödinger picture (S) and interaction picture (I)

$$|\psi_I(t)\rangle = \hat{U}_0(t)^+ |\psi_S(t)\rangle \quad \text{def. of transf. } S \to I$$

$$|\psi_S(t)\rangle = \hat{U}(t)|\psi_S(0)\rangle \quad \& \quad |\Psi_I(0)\rangle = |\psi_S(0)\rangle$$

$$\Rightarrow |\psi_I(t)\rangle = \hat{U}_0(t)^+ \hat{U}(t) |\psi_I(0)\rangle$$

$$\Rightarrow \underline{\hat{U}_I(t) = \hat{U}_0(t)^+ \hat{U}(t)}$$

Schrödinger eq. $\Rightarrow$

$$i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}(t)\hat{U}(t)$$

$$i\hbar \frac{d}{dt} \hat{U}_0(t) = \hat{H}_0 \hat{U}_0(t) \Rightarrow i\hbar \frac{d}{dt} \hat{U}_0(t)^+ = -\hat{U}_0(t)^+ \hat{H}_0$$

$$\Rightarrow i\hbar \frac{d}{dt} \hat{U}_I(t) = i\hbar \frac{d}{dt} \hat{U}_0(t)^+ \hat{U}(t) + \hat{U}_0(t)\, i\hbar \frac{d}{dt} \hat{U}(t)$$

$$= \hat{U}_0(t)^+ \hat{H}(t)\hat{U}(t) - \hat{U}_0(t) \hat{H}_0 \hat{U}(t)$$

$$= \hat{U}_0(t)^+ \hat{W}(t)\hat{U}(t)$$

$$\equiv \hat{H}_I(t)\hat{U}_I(t)$$

$$\Rightarrow \underline{\hat{H}_I(t) = \hat{U}_0^+(t)\hat{W}(t)\hat{U}_0(t)} \qquad \hat{U}_0(t) = e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

$$\hat{W} = A\hat{x} \sin\omega t = A\sqrt{\frac{\hbar}{2m\omega_0}}(\hat{a} + \hat{a}^+) \sin\omega t$$

$$\Rightarrow \hat{H}_I(t) = A\sqrt{\frac{\hbar}{2m\omega_0}}\, e^{i\omega_0 t \hat{a}^+\hat{a}}(\hat{a} + \hat{a}^+) e^{-i\omega_0 t \hat{a}^+\hat{a}} \sin\omega t$$

$$= \underline{A\sqrt{\frac{\hbar}{2m\omega_0}}\left(e^{-i\omega_0 t}\hat{a} + e^{i\omega_0 t}\hat{a}^+\right) \sin\omega t}$$

$$= \theta(t)^* \hat{a} + \theta(t)\hat{a}^+ \quad \text{with } \underline{\theta(t) = A\sqrt{\frac{\hbar}{2m\omega_0}}\, e^{i\omega_0 t} \sin\omega t}$$

d) Assume

$$\hat{U}_I = e^{\xi a^+ - \xi^* \hat{a}} \, e^{i\varphi} = e^{\xi \hat{a}^+} e^{-\xi^* \hat{a}} \, e^{i\varphi - \frac{1}{2}\xi^* \xi}$$

$$\Rightarrow \frac{d\hat{U}_I}{dt} = \dot{\xi} \, \hat{a}^+ e^{\xi \hat{a}^+} e^{-\xi^* \hat{a}} \, e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

$$- e^{\xi \hat{a}^+} \dot{\xi}^* \hat{a} \, e^{-\xi^* \hat{a}} \, e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

$$+ e^{\xi \hat{a}^+} e^{-\xi^* \hat{a}} \left(i\dot{\varphi} - \tfrac{1}{2}(\dot{\xi}^* \xi + \xi^* \dot{\xi}) \right) e^{i\varphi - \frac{1}{2}\xi^*\xi}$$

use $\quad e^{\xi \hat{a}^+} \hat{a} \, e^{-\xi \hat{a}^+} = \hat{a} - \xi$

$$\frac{d\hat{U}_I}{dt} = \left[(\dot{\xi} \hat{a}^+ - \dot{\xi}^* \hat{a}) + (i\dot{\varphi} + \tfrac{1}{2}(\dot{\xi}^* \xi - \xi^* \dot{\xi})) \right] \hat{U}_I(t)$$

Of the form

$$i\hbar \frac{d\hat{U}_I}{dt} = \hat{H}_I(t) \, \hat{U}_I(t)$$

if: 1) $\theta = i\hbar \dot{\xi} \quad \Rightarrow \quad \xi(t) = -\frac{i}{\hbar} \int_0^t \theta(t') dt'$

2) $\dot{\varphi} = \frac{i}{2} (\dot{\xi}^* \xi - \xi^* \dot{\xi})$

e) Note: $\hat{U}_I(t) = e^{i\varphi(t)} \, \hat{D}(\xi(t))$

Time evolution in the Schrödinger picture

$$|\psi(t)\rangle = \hat{U}_0(t) \, \hat{U}_I(t) \, |\psi(0)\rangle$$

$$= \hat{U}_0(t) \, e^{i\varphi(t)} \hat{D}(\xi(t)) \, |z_0\rangle$$

$$= e^{i\varphi} \, \hat{U}_0(t) \, \hat{D}(\xi) \, \hat{D}(z_0) \, |0\rangle$$

Product of displacements operators

$$\hat{D}(\xi) \, \hat{D}(z_0) = e^{\xi \hat{a}^+ - \xi^* \hat{a}} \, e^{z_0 \hat{a}^+ - z_0^* \hat{a}}$$

$$= e^{(\xi + z_0)\hat{a}^+ - (\xi + z_0)^* \hat{a} + \frac{1}{2}(\xi z_0^* - \xi^* z)}$$

$$= e^{\frac{1}{2}(\xi z_0^* - \xi^* z_0)} \, \hat{D}(z_0 + \xi)$$

Use results from a):

$$\hat{U}_0(t)\,|z\rangle = e^{-\frac{i}{2}\omega_0 t}\,|e^{-i\omega_0 t}z\rangle$$

$$\Rightarrow \quad |\psi(t)\rangle = \exp\left(i\varphi + \tfrac{1}{2}(\xi z_0^* - \xi^* z_0)\right)\hat{U}_0(t)\,|z_0 + \xi(t)\rangle$$

$$= \exp\left(i(\varphi - \tfrac{1}{2}\omega_0 t) + \tfrac{1}{2}(\xi z_0^* - \xi^* z_0)\right)|e^{-i\omega_0 t}(z_0 + \xi(t))\rangle$$

$$\Rightarrow \quad \gamma(t) = \varphi(t) - \tfrac{1}{2}\omega_0 t + \tfrac{1}{2i}\left(\xi(t) z_0^* - \xi(t)^* z_0\right)$$

$$z(t) = e^{-i\omega_0 t}(z_0 + \xi(t))$$

f) The function $\xi(t)$

$$\xi(t) = -\frac{i}{\hbar}\int_0^t \theta(t')\,dt'$$

with $\theta(t) = -\frac{i}{2}A\sqrt{\frac{\hbar}{2m\omega_0}}\left(e^{i(\omega_0+\omega)t} - e^{i(\omega_0-\omega)t}\right)$

$$\Rightarrow \quad \xi(t) = \frac{i}{2}A\sqrt{\frac{1}{2m\omega_0\hbar}}\left(\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} - \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right)$$

$$z(t) = z_0 e^{-i\omega_0 t} + \frac{i}{2}A\sqrt{\frac{1}{2m\omega_0\hbar}}\left(\frac{e^{i\omega t}-e^{-i\omega_0 t}}{\omega_0+\omega} - \frac{e^{-i\omega t}-e^{-i\omega_0 t}}{\omega_0-\omega}\right)$$

$$= \left(z_0 + i\frac{A}{\sqrt{2m\omega_0\hbar}}\frac{\omega}{\omega_0^2-\omega^2}\right)e^{-i\omega_0 t}$$

$$-\,i\frac{A}{\sqrt{2m\omega_0\hbar}}\frac{1}{\omega_0^2-\omega^2}\left(\omega\cos\omega t - i\omega_0\sin\omega t\right)$$

For the coherent state: $x(t) = \langle\psi(t)|\hat{x}|\psi(t)\rangle$ with $x(t) = \sqrt{2\hbar/m\omega_0}\,\mathrm{Re}\,z(t)$. $x(t)$ satisfies the same eq. of motion as the Heisenberg eq. of motion for $\hat{x}$. This is identical to the class. eq. of motion.

Can also be verified by explicit calculation.

FYS4110 Midtterminekesamen, høsten 2009

Løsninger

Oppgave 1

a) Egenverdier til $\hat{H}_0 = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \frac{1}{2}\hbar\omega_0\sigma_z$

Benytter $\hat{a}^\dagger\hat{a}\,|n,m\rangle = n|n,m\rangle$

$\sigma_z\,|n,m\rangle = 2m|n,m\rangle \qquad m = \pm\frac{1}{2}$

$\Rightarrow$ egenverdier: $\underline{E^0_{nm} = \hbar\,[(n+\frac{1}{2})\omega + m\omega_0]}$

Operatorene $\hat{a}\sigma_+$ og $\hat{a}^\dagger\sigma_-$ kobler (har matriseelementer) mellom tilstander med energiforskjell $\Delta E = \pm\hbar(\omega-\omega_0)$, mens operatorene $\hat{a}\sigma_-$ og $\hat{a}^\dagger\sigma_+$ kobler tilstander med energiforskjell $\Delta E = \pm\hbar(\omega+\omega_0)$. $\hat{H}_1$ blander sammen egentilstandene til $\hat{H}_0$ mer effektivt når energidifferansen er liten enn når den er stor. (Se f eks. uttrykk i perturbasjonsteori). Når $|\omega+\omega_0| \gg |\omega-\omega_0|$ er derfor betydningen av leddene som er strøket mye mindre enn betydningen av de som er beholdt.

b) Operatorene $\hat{a}\sigma_+$ og $\hat{a}\sigma_-$ kobler sammen par av tilstander $|n,-\frac{1}{2}\rangle$ og $|n-1,+\frac{1}{2}\rangle$ for $n = 1, 2, \dots$:

$\hat{a}\sigma_+\,|n,-\frac{1}{2}\rangle = \sqrt{n}\,|n-1,+\frac{1}{2}\rangle$

$\hat{a}^\dagger\sigma_-\,|n,-\frac{1}{2}\rangle = 0$

$\hat{a}\sigma_+\,|n-1,+\frac{1}{2}\rangle = 0$

$\hat{a}^\dagger\sigma_-\,|n-1,+\frac{1}{2}\rangle = \sqrt{n}\,|n,-\frac{1}{2}\rangle$

ingen kobling mellom $|n,-\frac{1}{2}\rangle$, $|n-1,+\frac{1}{2}\rangle$ og andre egentilstander til $\hat{H}_0$. Operatoren $\hat{H}_1$ kobler derfor også bare disse parene av tilstander.

Spesielt ; $n = 1$:

$$\langle 0, +\tfrac{1}{2} | \hat{H}_1 | 1, -\tfrac{1}{2} \rangle = \langle 1, -\tfrac{1}{2} | \hat{H}_1 | 0, +\tfrac{1}{2} \rangle = \tfrac{1}{2}\hbar\lambda$$

$$\langle 0, +\tfrac{1}{2} | \hat{H}_1 | 0, +\tfrac{1}{2} \rangle = \langle 1, -\tfrac{1}{2} | \hat{H}_1 | 1, -\tfrac{1}{2} \rangle = 0$$

Egenverdiligning i to-dimensjonalt underrom

$$\hat{H} | \psi \rangle = E | \psi \rangle \quad \text{med} \quad | \psi \rangle = c_1 | 0, +\tfrac{1}{2} \rangle + c_2 | 1, -\tfrac{1}{2} \rangle$$

på matriseform

$$\tfrac{1}{2}\hbar \begin{pmatrix} \omega + \omega_0 & \lambda \\ \lambda & 3\omega - \omega_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \omega_0 - \omega - \varepsilon & \lambda \\ \lambda & \omega - \omega_0 - \varepsilon \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

med $\varepsilon = 2\left(\frac{E}{\hbar} - \omega\right)$

Determinant-betingelse

$$\begin{vmatrix} \omega_0 - \omega - \varepsilon & \lambda \\ \lambda & \omega - \omega_0 - \varepsilon \end{vmatrix} = 0$$

$$\Rightarrow \quad \varepsilon^2 - (\omega - \omega_0)^2 - \lambda^2 = 0 \quad \Rightarrow \quad \varepsilon_\pm = \pm\sqrt{(\omega - \omega_0)^2 + \lambda^2} \equiv \pm \Omega$$

$$E_\pm = \hbar(\omega + \tfrac{1}{2}\varepsilon_\pm)$$

$$= \underline{\hbar(\omega \pm \tfrac{1}{2}\Omega)} = \underline{\hbar(\omega \pm \tfrac{1}{2}\sqrt{\Delta\omega^2 + \lambda^2})}$$

c) Koeffisienter c_1 og c_2

$$E_+ : \quad (\omega_0 - \omega - \varepsilon_+) c_{1+} + \lambda c_{2+} = 0$$

$$\Rightarrow \quad (\Delta\omega + \Omega) c_{1+} = \lambda c_{2+}$$

$$\Rightarrow \quad c_{1+} = N\lambda \ ; \quad c_{2+} = N(\Delta\omega + \Omega)$$

normalisering : $|c_{1+}|^2 + |c_{2+}|^2 = 1 \Rightarrow N = [(\Delta\omega + \Omega)^2 + \lambda^2]^{-1/2}$

Def: $c_{1+} = \cos\beta$, $c_{2+} = \sin\beta$

$$\Rightarrow \quad \cos\beta = \frac{\lambda}{\sqrt{(\Delta\omega+\Omega)^2+\lambda^2}} = \frac{\lambda}{\sqrt{2(\Delta\omega^2+\lambda^2+\Delta\omega\sqrt{\Delta\omega^2+\lambda^2}}} = \frac{\lambda}{\sqrt{2\Omega(\Omega+\Delta\omega)}}$$

$$\sin\beta = -\frac{\Delta\omega+\Omega}{\sqrt{(\Delta\omega+\Omega)^2+\lambda^2}} = -\frac{\Delta\omega+\sqrt{\Delta\omega^2+\lambda^2}}{\sqrt{2(\Delta\omega^2+\lambda^2+\Delta\omega\sqrt{\Delta\omega^2+\lambda^2}}} = -\sqrt{\frac{\Omega+\Delta\omega}{2\Omega}}$$

Egentilstand med egenverdi E_-

ortogonalitet $\langle\psi_+|\psi_-\rangle = 0 \Rightarrow c_{1+}^* c_{1-} + c_{2+}^* c_{2-} = 0$

$$\Rightarrow \quad c_{1-} = -c_{2+} = +\sin\beta$$

$$c_{2-} = c_{1+} = \cos\beta$$

(Entydig opp til multiplikasjon med en felles fasefaktor.)

d) Initialtilstand

$$|\psi(0)\rangle = |0,+\tfrac{1}{2}\rangle = \cos\beta\,|\psi_+\rangle + \sin\beta\,|\psi_-\rangle$$

Tidsutvikling

$$|\psi(t)\rangle = \cos\beta\, e^{-\frac{i}{\hbar}E_+t}|\psi_+\rangle + \sin\beta\, e^{-\frac{i}{\hbar}E_-t}|\psi_-\rangle$$

$$= (\cos^2\beta\, e^{-\frac{i}{\hbar}E_+t} + \sin^2\beta\, e^{-\frac{i}{\hbar}E_-t})\,|0,+\tfrac{1}{2}\rangle$$

$$- \cos\beta\sin\beta\,(e^{-\frac{i}{\hbar}E_+t} - e^{-\frac{i}{\hbar}E_-t})\,|1,-\tfrac{1}{2}\rangle$$

$$= c_1(t)\,|0,+\tfrac{1}{2}\rangle + c_2(t)\,|1,-\tfrac{1}{2}\rangle$$

koeffisienter

$$c_1(t) = e^{-i\omega t}(\cos^2\beta\, e^{-\frac{i}{2}\Omega t} + \sin^2\beta\, e^{\frac{i}{2}\Omega t})$$

$$= e^{-i\omega t}(\cos\tfrac{\Omega}{2}t - i\cos 2\beta\,\sin\tfrac{\Omega}{2}t)$$

$$= e^{-i\omega t}(\cos\tfrac{\Omega}{2}t + i\,\frac{\Delta\omega}{\Omega}\sin\tfrac{\Omega}{2}t$$

$$c_2(t) = \cos\beta\sin\beta\, e^{-i\omega t}(e^{\frac{i}{2}\Omega t} - e^{-\frac{i}{2}\Omega t}) = i\,e^{-i\omega t}\sin 2\beta\,\sin\tfrac{\Omega}{2}t$$

$$= -i\,e^{-i\omega t}\,\frac{\lambda}{\Omega}\sin\tfrac{\Omega}{2}t$$

e) $|c_2(t)|^2 = \sin^2 2\beta \, \sin^2 \frac{\Omega}{2} t$

$$= \frac{1}{2} \sin^2 2\beta \, (1 - \cos \Omega t)$$

$\cos \Omega t$ - periodisk funksjon med periode $T = \frac{2\pi}{\Omega} = \frac{2\pi}{\sqrt{\Delta\omega^2 + \lambda^2}}$

Maksimalverdi for $|c_2|^2$:

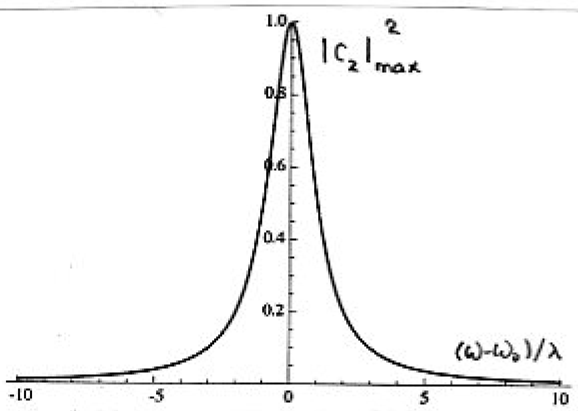
for $\sin^2 \frac{\Omega}{2} t = 1 \Rightarrow t = T(n + \frac{1}{2})$ n-heltall

$$|c_2|^2_{max} = \sin^2 2\beta = (2 \sin\beta \cos\beta)^2$$

$$2 \sin\beta \cos\beta = -2 \frac{\lambda}{\sqrt{2\Omega(\Omega + \Delta\omega)}} \sqrt{\frac{\Omega + \Delta\omega}{2\Omega}} = -\frac{\lambda}{\Omega}$$

$$\Rightarrow |c_2|^2_{max} = \frac{\lambda^2}{\Omega^2} = \frac{\lambda^2}{\Delta\omega^2 + \lambda^2} = \frac{\lambda^2}{(\omega - \omega_0)^2 + \lambda^2}$$

Som funksjon av ω_0, med ω og λ fast:



Resonans for $\omega_0 = \omega \Rightarrow |c_2|^2_{max} = 1$, størst mulig verdi.

f) Tetthetsoperator

$$\hat{\rho}(t) = |c_1(t)|^2 |0, +\tfrac{1}{2}\rangle\langle 0, +\tfrac{1}{2}| + |c_2(t)|^2 |1, -\tfrac{1}{2}\rangle\langle 1, -\tfrac{1}{2}|$$
$$+ c_1(t) c_2(t)^* |0, +\tfrac{1}{2}\rangle\langle 1, -\tfrac{1}{2}| + c_1(t)^* c_2(t) |1, -\tfrac{1}{2}\rangle\langle 0, +\tfrac{1}{2}|$$

Redusert tetthetsoperator for spinn

$$\hat{\rho}_s(t) = \sum_n \langle n | \hat{\rho}(t) | n \rangle$$
$$= |c_1(t)|^2 |+\tfrac{1}{2}\rangle\langle +\tfrac{1}{2}| + |c_2(t)|^2 |-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}|$$
$$= (1 - \sin^2 2\beta \sin^2 \tfrac{\Omega}{2} t) |+\tfrac{1}{2}\rangle\langle +\tfrac{1}{2}| + \sin^2 2\beta \sin^2 \tfrac{\Omega}{2} t \, |-\tfrac{1}{2}\rangle\langle -\tfrac{1}{2}|$$

Redusert posisjons-tetthetsmatrise

$$\hat{\rho}_P(t) = \sum_{m=-1/2}^{+1/2} \langle m|\hat{\rho}(t)|m\rangle$$

$$= |c_1(t)|^2 |0\rangle\langle 0| + |c_2(t)|^2 |1\rangle\langle 1|$$

$$= \underline{(1-\sin^2 2\beta \sin^2\tfrac{\Omega}{2}t)|0\rangle\langle 0| + \sin^2 2\beta \sin^2\tfrac{\Omega}{2}t\, |1\rangle\langle 1|}$$

g) Forventningsverdier

$$\langle\vec{\sigma}(t)\rangle = Tr_s(\vec{\sigma}\hat{\rho}_s(t))$$

$$= |c_1(t)|^2 \langle +\tfrac{1}{2}|\vec{\sigma}|+\tfrac{1}{2}\rangle + |c_2(t)|^2\langle -\tfrac{1}{2}|\vec{\sigma}|-\tfrac{1}{2}\rangle$$

$$= (|c_1(t)|^2 - |c_2(t)|^2)\,\vec{k}$$

$$= \underline{(1-2\sin^2 2\beta \sin^2\tfrac{\Omega}{2}t)\,\vec{k}}$$

$$\langle x(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, Tr_P((\hat{a}+\hat{a}^\dagger)\hat{\rho}_P(t)) \; \underline{= 0}$$

$$\langle x\vec{\sigma}(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, Tr((\hat{a}+\hat{a}^\dagger)\vec{\sigma}\hat{\rho}(t))$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\Big\{\langle 0,\tfrac{1}{2}|\hat{a}\,\sigma_+|1,-\tfrac{1}{2}\rangle c_1(t)c_2(t)^*(\vec{i}+i\vec{j})$$

$$+\langle 1,-\tfrac{1}{2}|\hat{a}\,\sigma_-|0,+\tfrac{1}{2}\rangle c_1(t)^*c_2(t)(\vec{i}-i\vec{j})$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\Big\{(c_1(t)c_2(t)^* + c_1(t)^*c_2(t))\,\vec{i} + i(c_1(t)c_2(t)^* - c_1(t)^*c_2(t))\,\vec{j}\Big\}$$

$$= \underline{-\sqrt{\frac{\hbar}{2m\omega}}\left(\sin 4\beta \sin^2\tfrac{\Omega}{2}t\;\vec{i} - \sin 2\beta\, \sin\Omega t\;\vec{j}\right)}$$

Partikkelen oscillerer i potensialet samtidig som spinnet presesserer rundt $\vec{B}$-feltet. Variasjonen i $\langle\vec{\sigma}(t)\rangle$ viser at energien oscillerer mellom spinn-energi og bevegelsesenergi. Tidsmidlet posisjon er $\langle x\rangle = 0$, mens $\langle x\vec{\sigma}(t)\rangle$ viser at oscillasjonene i x-koordinaten er korrelert med spinnbevegelsen.

Oppgave 2

a) Unitaritet

$$S_\lambda^\dagger = e^{\frac{1}{2}(\lambda \hat{a}^{\dagger} - \lambda^* \hat{a})} = S_\lambda^{-1} \quad \Rightarrow \quad S_\lambda^\dagger S_\lambda = \mathbb{1}$$

Transformasjon av senkeoperator

$$\hat{b}_\lambda = S_\lambda \hat{a} S_\lambda^\dagger = e^{\chi} \hat{a} e^{-\chi} \qquad \chi = \tfrac{1}{2}(\lambda^* \hat{a}^2 - \lambda \hat{a}^{\dagger 2})$$

$$= \hat{a} + [\chi, \hat{a}] + \frac{1}{2!}[\chi, [\chi, \hat{a}]] + \cdots$$

$$[\chi, \hat{a}] = -\tfrac{1}{2}\lambda [\hat{a}^{\dagger 2}, \hat{a}] = \lambda \hat{a}^{\dagger}$$

$$[\chi, \hat{a}^{\dagger}] = \tfrac{1}{2}\lambda^* [\hat{a}^2, \hat{a}^{\dagger}] = \lambda^* \hat{a}$$

$$\Rightarrow \quad \hat{b}_\lambda = \hat{a} + \lambda \hat{a}^{\dagger} + \tfrac{1}{2}|\lambda|^2 a + \frac{1}{3!}\lambda |\lambda|^2 \hat{a}^{\dagger} +$$

$$= \hat{a}\left(1 + \frac{1}{2!}|\lambda|^2 + \frac{1}{4!}|\lambda|^4 + \cdots\right)$$

$$+ \frac{\lambda}{|\lambda|}\hat{a}^{\dagger}\left(|\lambda| + \frac{1}{3!}|\lambda|^3 + \cdots\right)$$

$$= \underline{\cosh|\lambda|\, \hat{a} + \frac{\lambda}{|\lambda|}\sinh|\lambda|\, \hat{a}^{\dagger}}$$

$$\Rightarrow \quad \underline{\hat{b}_\lambda^{\dagger} = \cosh|\lambda|\, \hat{a}^{\dagger} + \frac{\lambda^*}{|\lambda|}\sinh|\lambda|\, \hat{a}}$$

$$[\hat{a}, \hat{a}^{\dagger}] = \mathbb{1} \quad \Rightarrow$$

$$[\hat{b}_\lambda, \hat{b}_\lambda^{\dagger}] = [S_\lambda \hat{a} S_\lambda^\dagger, S_\lambda \hat{a}^{\dagger} S_\lambda^\dagger]$$

$$= S_\lambda [\hat{a}, \hat{a}^{\dagger}] S_\lambda^\dagger = S_\lambda S_\lambda^\dagger = \underline{\mathbb{1}}$$

Samme kommutator

b) Egenvektor til $\hat b_\lambda$,

$$\hat b_\lambda |z,\lambda\rangle = \hat b_\lambda S_\lambda |z\rangle$$

$$= S_\lambda S_\lambda^+ \hat b_\lambda S_\lambda |z\rangle$$

$$\hat b_\lambda = S_\lambda \hat a S_\lambda^+ \Rightarrow \hat a = S_\lambda^+ \hat b_\lambda S_\lambda$$

$$\hat b_\lambda |z,\lambda\rangle = S_\lambda \hat a |z\rangle$$

koherent tilstand: $\hat a |z\rangle = z|z\rangle$

$$\Rightarrow \hat b_\lambda |z,\lambda\rangle = z S_\lambda |z\rangle = \underline{z|z,\lambda\rangle}$$

$|z,\lambda\rangle$ egentilstand, med z som egenverdi

c) $\lambda = \lambda^*$ reell:

$$\Rightarrow \hat b_\lambda = \cosh\lambda\, \hat a + \sinh\lambda\, \hat a^+$$
$$\hat b_\lambda^+ = \cosh\lambda\, \hat a^+ + \sinh\lambda\, \hat a$$

$$\hat x = \sqrt{\frac{\hbar}{2m\omega}}(\hat a + \hat a^+) \qquad \hat p = -i\sqrt{\frac{\hbar m\omega}{2}}(\hat a - i\hat a^+)$$

$$S_\lambda \hat x S_\lambda^+ = \sqrt{\frac{\hbar}{2m\omega}}(\hat b_\lambda + \hat b_\lambda^+)$$
$$= (\cosh\lambda + \sinh\lambda)\sqrt{\frac{\hbar}{2m\omega}}(\hat a + \hat a^+)$$
$$= \underline{e^\lambda \hat x}$$

$$S_\lambda \hat p S_\lambda^+ = -i\sqrt{\frac{\hbar}{2m\omega}}(\hat b_\lambda - \hat b_\lambda^+)$$
$$= (\cosh\lambda - \sinh\lambda)\left(-i\sqrt{\frac{\hbar}{2m\omega}}(\hat a - \hat a^+)\right)$$
$$= \underline{e^{-\lambda}\hat p}$$

dvs $S_\lambda \hat x S_\lambda^+ = d\,\hat x$, $S_\lambda \hat p S_\lambda^+ = \frac{1}{d}\hat p$, med $\underline{d = e^\lambda}$

$$\Rightarrow \Delta x_{z\lambda}^2 = \langle z|(S_\lambda^+ \hat x S_\lambda)^2|z\rangle - \langle z|S_\lambda^+ \hat x S_\lambda|z\rangle^2 = \frac{1}{d^2}\Delta x_z^2 \,;\quad \Delta p_{z\lambda}^2 = d^2 \Delta p_z^2$$

$\Rightarrow \quad \Delta x_{z\lambda}\, \Delta p_{z\lambda} = \Delta x_z \Delta p_z = \frac{\hbar}{2}$, samme som for koherent tilstand

d) Presset grunntilstand

$$|0,\lambda\rangle = S_\lambda |0\rangle$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\tfrac{1}{2}(\lambda^* \hat{a}^2 - \lambda \hat{a}^{\dagger 2})\right)^n |0\rangle$$

S_λ inneholder bare kvadratiske operatorer i $\hat{a}$ og $\hat{a}^\dagger$. Kan derfor bare heve n med et <u>like</u> antall trinn fra $n=0$.

Benytter $\hat{b}_\lambda |0,\lambda\rangle = 0$ fra b)

og $\hat{b}_\lambda = \cosh|\lambda|\,\hat{a} + \frac{\lambda}{|\lambda|}\sinh|\lambda|\,\hat{a}^\dagger$ fra a)

$$\hat{b}_\lambda \sum_n c_n |2n\rangle = 0 \Rightarrow$$

$$\cosh|\lambda| \sum_n c_n \sqrt{2n}\,|2n-1\rangle + \frac{\lambda}{|\lambda|}\sinh|\lambda| \sum_n c_n \sqrt{2n+1}\,|2n+1\rangle = 0$$

$$\Rightarrow \sum_n \left(\cosh|\lambda|\sqrt{2n}\,c_n + \frac{\lambda}{|\lambda|}\sinh|\lambda|\sqrt{2n-1}\,c_{n-1}\right)|2n-1\rangle = 0$$

Hver koeffisient i rekken må forsvinne:

$$\cosh|\lambda|\sqrt{2n}\,c_n + \frac{\lambda}{|\lambda|}\sinh|\lambda|\sqrt{2n-1}\,c_{n-1} = 0$$

$$\Rightarrow \quad c_n = -\frac{\lambda}{|\lambda|}\tanh|\lambda|\sqrt{\frac{2n-1}{2n}}\,c_{n-1}$$

$$= \left(-\frac{\lambda}{|\lambda|}\tanh|\lambda|\right)^n \sqrt{\frac{(2n-1)(2n-3)\cdots 1}{2n(2n-2)\cdots 2}}\,c_0$$

$$= \left(-\frac{\lambda}{|\lambda|}\tanh|\lambda|\right)^n \frac{\sqrt{(2n)!}}{2^n\,n!}\,c_0$$

$$= \left(-\frac{\lambda}{2|\lambda|}\tanh|\lambda|\right)^n \frac{\sqrt{(2n)!}}{n!}\,c_0$$

Normalisering $\quad \sum_n |c_n|^2 = 1$

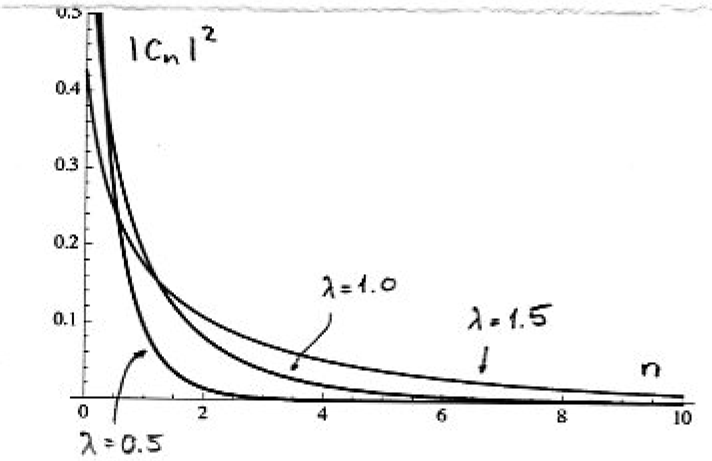
$$\Rightarrow |c_0|^{-2} = \sum_{n=0}^{\infty} |a|^{2n} \frac{2n!}{(n!)^2} \qquad |a| = \tfrac{1}{2}\tanh|\lambda|$$

$$= \frac{1}{\sqrt{1-4|a|^2}} = \frac{1}{\sqrt{1-\tanh^2|\lambda|}} = \cosh|\lambda|$$

koeffisienter

$$c_n = \frac{1}{\sqrt{\cosh|\lambda|}} \left(-\frac{\lambda}{2|\lambda|}\tanh|\lambda|\right)^n \frac{\sqrt{(2n)!}}{n!}$$

e) Plot av $|c_n|^2$



$|c_n|^2$ faller monotont av med økende n

For $\lambda = 0$ er bare $|c_0|^2 \neq 0$ $(=1)$

når $\lambda \neq 0$ er $|c_n|^2 \neq 0$ for alle n,

og jo større λ desto langsommere avtar $|c_n|^2$ med økende n

f) Benytter at $|z, \lambda\rangle$ er egentilstand for $\hat{b}_\lambda$, for generell λ. Studerer

$$\hat{b}_\lambda e^{-\frac{i}{\hbar}\hat{H}t} |z_0, \lambda_0\rangle \qquad \text{for uspesifisert } \lambda$$

$$= e^{-\frac{i}{\hbar}\hat{H}t} e^{\frac{i}{\hbar}\hat{H}t} (\cosh|\lambda|\,\hat{a} + \frac{\lambda}{|\lambda|}\sinh|\lambda|\,\hat{a}^+) e^{-\frac{i}{\hbar}\hat{H}t} |z_0, \lambda\rangle$$

$$e^{\frac{i}{\hbar}\hat{H}t}\,\hat{a}\,e^{-\frac{i}{\hbar}\hat{H}t} = e^{i\hat{a}^+\hat{a}t}\,\hat{a}\,e^{-i\hat{a}^+\hat{a}t} = e^{-i\omega t}\,\hat{a}$$

$$e^{\frac{i}{\hbar}\hat{H}t}\,\hat{a}^+ e^{-\frac{i}{\hbar}\hat{H}t} = e^{i\omega t}\,\hat{a}^+$$

$$\Rightarrow \hat{b}_\lambda e^{-\frac{i}{\hbar}\hat{H}t}|z_0,\lambda\rangle = e^{-\frac{i}{\hbar}\hat{H}t} e^{-i\omega t}\left(\cosh|\lambda|\hat{a} + \frac{\lambda e^{2i\omega t}}{|\lambda|}\sinh|\lambda|\right)|z_0,\lambda\rangle$$

$$= e^{-i\omega t} e^{-\frac{i}{\hbar}\hat{H}t}\hat{b}_{(\lambda e^{2i\omega t})}|z_0,\lambda_0\rangle$$

Uttrykket gjelder for vilkårlig valgt λ.

Velger nå $\lambda e^{2i\omega t} = \lambda_0$, dvs $\lambda = \lambda_0 e^{-2i\omega t}$ $\Rightarrow$

$$\hat{b}_{(\lambda e^{2i\omega t})}|z_0,\lambda\rangle = \hat{b}_{\lambda_0}|z_0,\lambda_0\rangle = z_0|z_0,\lambda\rangle$$

$$\Rightarrow \hat{b}_{(\lambda_0 e^{-2i\omega t})} e^{-\frac{i}{\hbar}\hat{H}t}|z_0,\lambda\rangle = e^{-i\omega t} z_0 e^{-\frac{i}{\hbar}\hat{H}t}|z_0,\lambda\rangle$$

dvs $e^{-\frac{i}{\hbar}\hat{H}t}|z_0,\lambda\rangle$ er egenvektor for $\hat{b}_{(\lambda_0 e^{-2i\omega t})}$

med egenverdi $z_0 e^{-i\omega t}$

$$\Rightarrow \underline{e^{-\frac{i}{\hbar}\hat{H}t}|z_0,\lambda\rangle = e^{i\alpha(t)}|z_0 e^{-i\omega t}, \lambda_0 e^{-2i\omega t}\rangle}$$

$\alpha(t)$ ubestemt kompleks fase

Tidsutvikling, presset tilstand på formen $e^{i\alpha(t)}|z(t),\lambda(t)\rangle$

med $z(t) = z_0 e^{-i\omega t}$ og $\lambda(t) = \lambda_0 e^{-2i\omega t}$

g) $z_0 = 0$

$\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$; $\hat{x}$ og $\hat{p}$ er lineære i $\hat{a}$ og $\hat{a}^+$

$\Rightarrow$ alle matriseelementer mellom tilstandene $|2n\rangle$ forsvinner.

$$\Rightarrow \Delta x^2 = \langle\hat{x}^2\rangle = \frac{\hbar}{2m\omega}\langle(\hat{a}+\hat{a}^+)^2\rangle$$

$$\Delta p^2 = \langle\hat{p}^2\rangle = -\frac{\hbar m\omega}{2}\langle(\hat{a}-\hat{a}^+)^2\rangle$$

benytter:

$$\hat{a} + \hat{a}^+ = c\, b_\lambda - c^* b_\lambda^+ \quad ; \quad c = \cosh|\lambda| - \frac{\lambda^*}{|\lambda|}\sinh|\lambda|$$

$$\hat{a} - \hat{a}^+ = d\, b_\lambda - d^* b_\lambda^+ \quad ; \quad d = \cosh|\lambda| + \frac{\lambda^*}{|\lambda|}\sinh|\lambda|$$

$$\langle(\hat{a}+\hat{a}^\dagger)^2\rangle = c^2\langle 0,\lambda|\hat{b}_\lambda^2|0,\lambda\rangle + c^{*2}\langle 0,\lambda|\hat{b}_\lambda^{\dagger 2}|0,\lambda\rangle$$
$$+ cc^*\langle 0,\lambda|\hat{b}_\lambda^\dagger\hat{b}_\lambda + \hat{b}_\lambda^\dagger b_\lambda|0,\lambda\rangle$$

$$\lambda = \lambda(t) = \lambda_0 e^{-2i\omega t}$$

benytter $\langle\hat{b}_\lambda^2\rangle = \langle\hat{b}_\lambda^{\dagger 2}\rangle = \langle\hat{b}_\lambda^\dagger b_\lambda\rangle = 0$

$$\langle\hat{b}_\lambda\hat{b}_\lambda^\dagger\rangle = 1$$

$$\Rightarrow \langle(\hat{a}+\hat{a}^\dagger)^2\rangle = |c|^2 = \cosh 2|\lambda| - \sinh 2|\lambda|\,\frac{\mathrm{Re}\,\lambda}{|\lambda|}$$
$$= \cosh 2\lambda_0 - \sinh 2\lambda_0 \cos 2\omega t$$

Tilsvarende

$$\langle(\hat{a}-\hat{a}^\dagger)^2\rangle = |d|^2 = \cosh 2\lambda_0 + \sinh 2\lambda_0 \cos 2\omega t$$

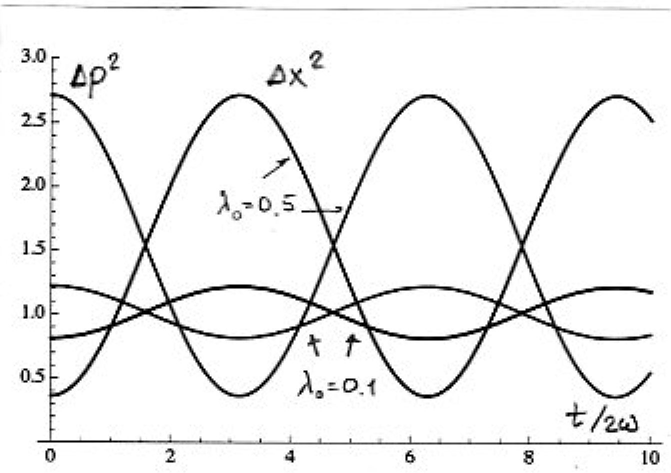
$$\underline{\Delta x^2 = \frac{\hbar}{2m\omega}(\cosh 2\lambda_0 - \sinh 2\lambda_0 \cos 2\omega t)}$$

$$\underline{\Delta p^2 = \frac{\hbar m\omega}{2}(\cosh 2\lambda_0 + \sinh 2\lambda_0 \cos 2\omega t)}$$

Plot av Δx^2 og Δp^2, normalisert med faktorene:

$\Delta x^2 \to \frac{2m\omega}{\hbar}\Delta x^2$

$\Delta p^2 \to \frac{2}{\hbar m\omega}\Delta p^2$



Variansene Δx^2 og Δp^2 varierer periodisk i t, med periode $\frac{\pi}{\omega}$; de varierer i motfase. Amplituden i oscillasjonene øker med λ_0.

Midttermineksamen, FYS 4110, høsten 2010
Løsninger

———————————————————————————-

OPPGAVE 1

a) Hamiltonoperatoren i $\{|\psi_L\rangle, |\psi_R\rangle\}$ basis er

$$H = \begin{pmatrix} E_0 & \lambda \\ \lambda & E_0 \end{pmatrix} \tag{1}$$

Egenverdiene E er bestemt av ligningen,

$$\begin{vmatrix} E_0 - E & \lambda \\ \lambda & E_0 - E \end{vmatrix} = 0 \quad \Rightarrow \quad (E - E_0)^2 - \lambda^2 = 0 \tag{2}$$

Løsninger

$$E_0^{\pm} = E_0 \pm \lambda \tag{3}$$

Egenvektorer på matriseform

$$\psi_0^{\pm} = \begin{pmatrix} \alpha_0^{\pm} \\ \beta_0^{\pm} \end{pmatrix}, \quad |\alpha_0^{\pm}|^2 + |\beta_0^{\pm}|^2 = 1 \tag{4}$$

Koeffisientene er bestemt av egenverdiligningen

$$\begin{aligned} \begin{pmatrix} E_0 & \lambda \\ \lambda & E_0 \end{pmatrix} \begin{pmatrix} \alpha_0^{\pm} \\ \beta_0^{\pm} \end{pmatrix} &= E_0^{\pm} \begin{pmatrix} \alpha_0^{\pm} \\ \beta_0^{\pm} \end{pmatrix} \\ &\Rightarrow \\ (E_0 - E^{\pm}_{=})\alpha_0^{\pm} &= -\lambda \beta_0^{\pm} \\ &\Rightarrow \\ \alpha_0^{\pm} = \pm \beta_0^{\pm} &= \frac{1}{\sqrt{2}} \end{aligned} \tag{5}$$

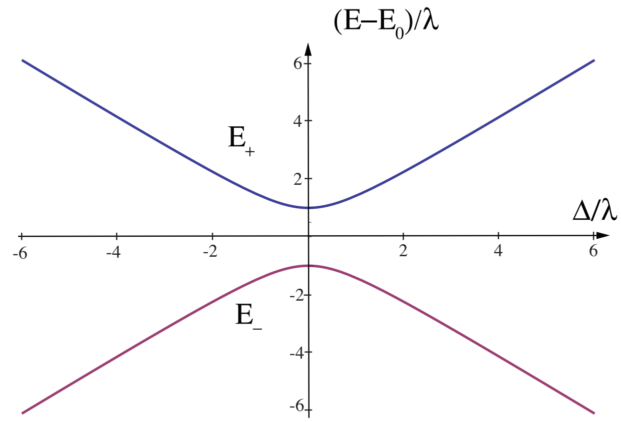
I braket-formulering

$$|\psi_0^{\pm}\rangle = \frac{1}{\sqrt{2}}(|\psi_L\rangle \pm |\psi_R\rangle) \tag{6}$$

Egenvektorene er den symmetriske og antisymmetriske superposisjon av $|\psi_L\rangle$ og $|\psi_R\rangle$. Den antisymmetriske superposisjon har lavest energi. Kan forstås ved at den har lavere sannsynlighet for at N-atomet befinner seg i potensialbarrieren hvor den potensielle energien er høyere.

b) Ny egenverdiligning

$$\begin{vmatrix} E_0 + \Delta - E & \lambda \\ \lambda & E_0 - \Delta - E \end{vmatrix} = 0 \quad \Rightarrow \quad (E - E_0)^2 = \lambda^2 + \Delta^2 \tag{7}$$

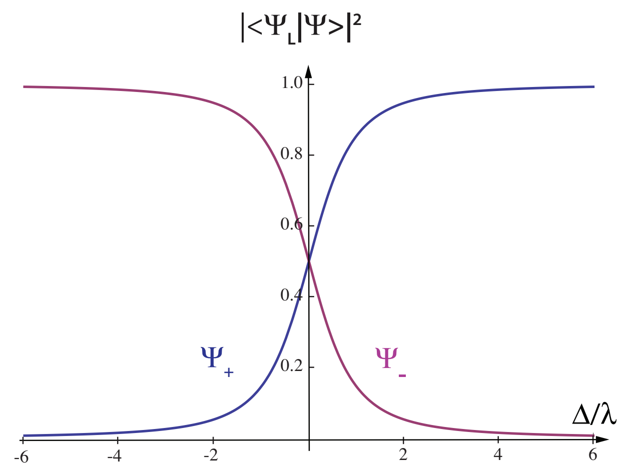


Løsninger

$$E_{\pm} = E_0 \pm \sqrt{\lambda^2 + \Delta^2} \tag{8}$$

c) Egenvektorer, matriseelementer

$$\begin{aligned}(E_0 + \Delta - E_{\pm})\alpha_{\pm} + \lambda\beta_{\pm} &= 0 \quad \Rightarrow \\ (\Delta \mp \sqrt{\lambda^2 + \Delta^2})\alpha_{\pm} + \lambda\beta_{\pm} &= 0\end{aligned} \tag{9}$$



Normerte løsninger

$$\begin{aligned}\alpha_{\pm} &= \frac{1}{\sqrt{2\sqrt{\lambda^2+\Delta^2}}}\sqrt{\sqrt{\lambda^2+\Delta^2} \pm \Delta} \\ \beta_{\pm} &= \pm\frac{1}{\sqrt{2\sqrt{\lambda^2+\Delta^2}}}\sqrt{\sqrt{\lambda^2+\Delta^2} \mp \Delta}\end{aligned} \tag{10}$$

Tilstander på braket-form

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2\sqrt{\lambda^2+\Delta^2}}}(\sqrt{\sqrt{\lambda^2+\Delta^2}\pm\Delta}\,|\psi_L\rangle \pm \sqrt{\sqrt{\lambda^2+\Delta^2}\mp\Delta}\,|\psi_R\rangle) \tag{11}$$

Overlapp

$$|\langle\psi_L|\psi_\pm\rangle|^2 = \frac{1}{2}(1\pm\frac{\Delta}{\sqrt{\lambda^2+\Delta^2}}) \tag{12}$$

Avoided crossing: Når Δ øker og passerer$\Delta = 0$ vil energinivåene nærme seg hverandre men unngår en direkte krysning ved en effektiv frastøtning mellom nivåene. Den minste avstanden er bestemt av λ. Tilstandsvektorene til de to nivåene byttes om nær dette punktet slik at grunntilstanden $|\psi_-\rangle$ svarer til $|\psi_L\rangle$ for stor negativ Δ og til $|\psi_R\rangle$ for stor positiv Δ.

d) Hamiltonoperator og tilstander i $\{|\psi_L\rangle, |\psi_R\rangle\}$ basis,

$$\hat{H} = \begin{pmatrix} E_0+\Delta & \lambda \\ \lambda & E_0-\Delta \end{pmatrix}, \quad \psi_0^\pm = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \tag{13}$$

Matriseelementer til $\hat{H}$ i $|\psi_0^\pm\rangle$ basis

$$\begin{aligned} \psi_0^{\pm\dagger}\hat{H}\psi_0^\pm &= \frac{1}{2}(1 \ \pm 1)\begin{pmatrix} E_0+\Delta & \lambda \\ \lambda & E_0-\Delta \end{pmatrix}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = E_0 \pm \lambda \\ \psi_0^{\pm\dagger}\hat{H}\psi_0^\mp &= \frac{1}{2}(1 \ \pm 1)\begin{pmatrix} E_0+\Delta & \lambda \\ \lambda & E_0-\Delta \end{pmatrix}\begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} = \Delta \end{aligned} \tag{14}$$

Det gir følgende matriseform for H i $|\psi_\pm\rangle$ basis,

$$\hat{H} = \begin{pmatrix} E_0+\lambda & \Delta \\ \Delta & E_0-\lambda \end{pmatrix} = E_0\mathbb{1} + \lambda\sigma_z + \Delta\sigma_x \tag{15}$$

og i det oscillerende elektriske felt, hvor $\Delta = \Delta_0\cos\omega t$, blir Hamiltonoperatoren

$$\hat{H} = E_0\mathbb{1} + \lambda\sigma_z + \Delta_0\cos\omega t\sigma_x \tag{16}$$

e) I den roterende bølge-tilnærmelsen får H følgende form

$$\begin{aligned} \hat{H} &= E_0\mathbb{1} + \lambda\sigma_z + \frac{1}{2}\Delta_0(e^{i\omega t}\sigma_- + e^{-i\omega t}\sigma_+) \\ &= E_0\mathbb{1} + \lambda\sigma_z + \frac{1}{2}\Delta_0(\cos\omega t\sigma_x + \sin\omega t\sigma_y) \end{aligned} \tag{17}$$

Den har samme form som Hamiltonoperatoren for et spinn-1/2-system i et konstant magnetfelt langs z-aksen superponert med et roterende magnetfelt i xy-planet. I forelesningsnotatene er Hamiltonoperatoren

$$\hat{H} = \frac{1}{2}\omega_0\hbar\sigma_z + \frac{1}{2}\omega_1\hbar(\cos\omega t\sigma_x + \sin\omega t\sigma_y) \tag{18}$$

hvor ω_0 er proporsjonal med styrken på det konstante feltet og ω_1 er proporsjonal med styrken på det roterende feltet. Sammenligningen av uttrykkene gir relasjonene

$$\lambda = \frac{1}{2}\omega_0\hbar, \quad \Delta_0 = \omega_1\hbar \tag{19}$$

I det følgende benyttes disse identitetene. Hamiltonoperatoren (17) har også et konstantledd $E_0 \mathbb{1}$, men dette er ikke av betydning for tidsutviklingen av systemet, siden den bare bidrar med en felles fasefaktor for alle tilstandene. I det følgende settes $E_0 = 0$.

Hamiltonoperatoren transformeres til tidsuavhengig form med den unitære, tidsavhengige transformasjonen

$$\hat{T}(t) = e^{\frac{i}{2}\omega t \sigma_z} \tag{20}$$

Den transformerte $\hat{H}$ blir

$$\begin{aligned} \hat{H}_{\hat{T}} &= \hat{T}(t)\hat{H}\hat{T}(t)^\dagger + i\hbar \frac{d\hat{T}}{dt}\hat{T}(t) \\ &= \frac{1}{2}\hbar\Omega(\cos\theta\sigma_z + \sin\theta\sigma_x) \end{aligned} \tag{21}$$

hvor

$$\Omega = \sqrt{(\omega - \omega_0)^2 + \omega_1^2} = \frac{1}{\hbar}\sqrt{(\omega\hbar - 2\lambda)^2 + \Delta_0^2} \tag{22}$$

er Rabifrekvensen og hvor θ er bestemt ved ligningene

$$\begin{aligned} \cos\theta &= \frac{\omega_0 - \omega}{\Omega} = \frac{2\lambda - \Delta_0}{\sqrt{(\omega\hbar - 2\lambda)^2 + \Delta_0^2}} \\ \sin\theta &= \frac{\omega_1}{\Omega} = \frac{\Delta_0}{\sqrt{(\omega\hbar - 2\lambda)^2 + \Delta_0^2}} \end{aligned} \tag{23}$$

Resonansfrekvensen er

$$\omega_0 = 2\lambda/\hbar \tag{24}$$

Tidsutviklingsoperatoren i det transformerte bildet er

$$\hat{\mathcal{U}}_T(t) = \cos(\frac{\Omega}{2}t)\mathbb{1} - i\sin(\frac{\Omega}{2}t)(\cos\theta\sigma_z + \sin\theta\sigma_x) \tag{25}$$

I Schrödingerbildet

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}\omega t\sigma_z}\hat{\mathcal{U}}_T(t) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{26}$$

med matriseelementer

$$\begin{aligned} A &= (\cos(\frac{\Omega}{2}t) - i\cos\theta\sin(\frac{\Omega}{2}t))e^{-\frac{i}{2}\omega t} \\ D &= (\cos(\frac{\Omega}{2}t) + i\cos\theta\sin(\frac{\Omega}{2}t))e^{\frac{i}{2}\omega t} \\ B &= -i\sin\theta\sin(\frac{\Omega}{2}t))e^{-\frac{i}{2}\omega t} \\ C &= -i\sin\theta\sin(\frac{\Omega}{2}t))e^{\frac{i}{2}\omega t} \end{aligned} \tag{27}$$

(For detaljerte mellomregninger refereres til forelesningsnotatene.)

f) Tilstander i $|\psi_0^\pm\rangle$ basis,

$$|\psi_L\rangle = \frac{1}{\sqrt{2}}(|\psi_0^+\rangle + |\psi_0^-\rangle)\,, \quad |\psi_R\rangle = \frac{1}{\sqrt{2}}(|\psi_0^+\rangle - |\psi_0^-\rangle) \tag{28}$$

På matriseform

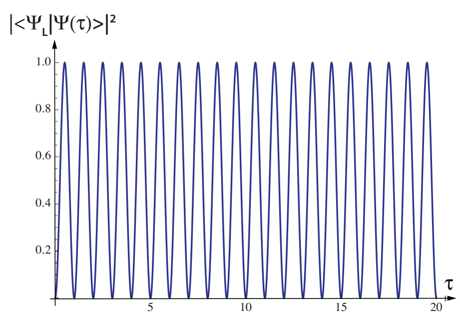
$$\psi_L = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\,, \quad \psi_R = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{29}$$

Overlapp

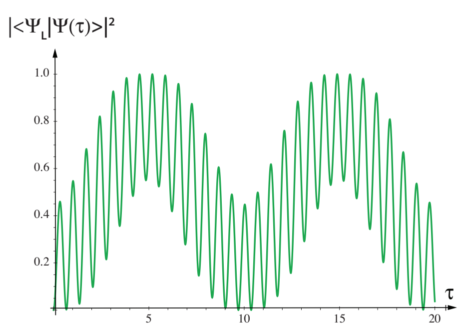
$$\begin{aligned}\langle\psi_R|\psi(t)\rangle &= \langle\psi_R|\hat{\mathcal{U}}(t)|\psi_L\rangle \\ &= \frac{1}{2}(1\ \ -1)\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \\ &= \frac{1}{2}((A-D)+(B-C))\end{aligned} \tag{30}$$

Innsatt for A, B, C, D,

$$\langle\psi_R|\psi(t)\rangle = -[\sin\theta\sin(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t) + i\{\cos(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t) + \cos\theta\sin(\frac{\Omega}{2}t)\cos(\frac{\omega}{2}t)\}] \tag{31}$$



ω = ω₀



$\omega = \omega_0/10$

g) Kvadrert uttrykk

$$
\begin{aligned}
|\langle\psi_R|\psi(t)\rangle|^2 &= [\sin\theta\sin(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t)]^2 + [\cos(\frac{\Omega}{2}t)\sin(\frac{\omega}{2}t) + \cos\theta\sin(\frac{\Omega}{2}t)\cos(\frac{\omega}{2}t)\}]^2 \\
&= \frac{1}{2}[1 - \cos\omega t + \cos^2\theta(1-\cos\Omega t)\cos\omega t + \cos\theta\sin\Omega t\sin\omega t] \qquad (32)
\end{aligned}
$$

Plot av funksjonen $|\langle\psi_R|\psi(t)\rangle|^2$ med $\tau = 2\pi\lambda t$ som tidskoordinat: De to figurene svarer til $\omega = \omega_0 = 2\lambda/\hbar$ og $\omega = \omega_0/10 = \lambda/5\hbar$. I begge tilfeller er $\omega_1 = \Delta_0/\hbar = 2\lambda/\hbar = \omega_0$.
Kommentar:
Ved resonans er oscillasjonene rene sinus-oscillasjonere med sirkelfrekvens ω_0. Det er det samme som når det periodiske feltet er slått av. Det er lett å sjekke av uttrykkene ovenfor at det oscillerende feltet ved resonans bare påvirker fasen til $\langle\psi_R|\psi(t)\rangle$. Ved $\omega = \omega_0/10$ er svingningene modulert av en langsommere oscillasjon som svarer omtrent til frekvensen ω. Den raskere frekvensen er også noe påvirket av oscillasjonene til det elektriske feltet. Uttrykket ovenfor viser at funksjonen $|\langle\psi_R|\psi(t)\rangle|^2$ er en lineær kombinasjon av tre periodiske funsjoner med frekvenser ω, $\Omega - \omega$ og $\Omega + \omega$.

OPPGAVE 2

a) Hamiltonoperator

$$
\begin{aligned}
\hat{H} &= \omega(\hat{S}_{1z} + \hat{S}_{2z}) + \frac{\alpha}{\hbar}[(\hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2)^2 - (\hat{\mathbf{S}}_1^2 + \hat{\mathbf{S}}_2^2)] \\
&= \omega\hat{S}_z + \frac{\alpha}{\hbar}[\hat{\mathbf{S}}^2 - \frac{3}{2}\hbar^2\mathbb{1}] \qquad (33)
\end{aligned}
$$

Egenverdier og egenvektorer

$$
\hat{H}|s,m\rangle = \left((s(s+1) - \frac{3}{2})\alpha + m\omega\right)\hbar\;|s,m\rangle \qquad (34)
$$

for de aktuelle tilstandene

$$
\begin{aligned}
\hat{H}|1,1\rangle &= (\frac{1}{2}\alpha + \omega)\hbar\;|1,1\rangle \\
\hat{H}|1,0\rangle &= \frac{1}{2}\alpha\hbar\;|1,0\rangle \\
\hat{H}|1,-1\rangle &= (\frac{1}{2}\alpha - \omega)\hbar\;|1,1\rangle \\
\hat{H}|1,1\rangle &= -\frac{3}{2}\alpha\hbar\;|1,1\rangle
\end{aligned} \qquad (35)
$$

b) Initialtilstand

$$
\begin{aligned}
\hat{\rho}(0) &= |\psi(0)\rangle\langle\psi(0)| \\
&= \frac{1}{2}(|++\rangle\langle++| \;+\; |+-\rangle\langle+-| \;+\; |++\rangle\langle+-| \;+\; |+-\rangle\langle++|) \qquad (36)
\end{aligned}
$$

Tilstanden er ren siden kan uttrykkes ved en enkelt tilstandsvektor. Den er ukorrelert siden den kan skrives som en produktvektor,

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|+\rangle \otimes (|+\rangle + |-\rangle) \tag{37}$$

Det er derfor ingen klassisk korrelasjon eller kvantemekanisk sammenfiltring mellom delsystemene. Redusert tetthetsoperator for spinn 1

$$\begin{aligned} \hat{\rho}_1(0) &= \mathrm{Tr}_2\,\hat{\rho}(0) = |+\rangle\langle+| = \frac{1}{2}(\mathbb{1} + \sigma_z) \\ &\Rightarrow \quad \mathbf{r}_1 = \mathbf{k} \end{aligned} \tag{38}$$

Redusert tetthetsoperator for spinn 2

$$\begin{aligned} \hat{\rho}_2(0) &= \mathrm{Tr}_1\,\hat{\rho}(0) \\ &= \frac{1}{2}(|+\rangle\langle+| \;+\; |-\rangle\langle-| \;+\; |+\rangle\langle-| \;+\; |-\rangle\langle+|) \\ &= \frac{1}{2}(\mathbb{1} + \sigma_x) \\ &\Rightarrow \quad \mathbf{r}_2 = \mathbf{i} \end{aligned} \tag{39}$$

c) Initialtilstand

$$\begin{aligned} |\psi(0)\rangle &= \frac{1}{\sqrt{2}}(|++\rangle + \frac{1}{2}(|+-\rangle + |-+\rangle) + \frac{1}{2}(|+-\rangle - |-+\rangle)) \\ &= \frac{1}{\sqrt{2}}(|1,1\rangle + \frac{1}{2}|1,0\rangle + \frac{1}{2}|0,0\rangle) \end{aligned} \tag{40}$$

Tidsutvikling

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}(e^{-i(\frac{1}{2}\alpha+\omega)t}\,|1,1\rangle + \frac{1}{2}e^{-i\frac{1}{2}\alpha t}\,|+-\rangle + \frac{1}{2}e^{i\frac{3}{2}\alpha t}\,|0,0\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-i(\frac{1}{2}\alpha+\omega)t}\,|++\rangle + \frac{1}{2}(e^{-i\frac{1}{2}\alpha t} + e^{i\frac{3}{2}\alpha t})\,|+-\rangle + \frac{1}{2}(e^{-i\frac{1}{2}\alpha t} - e^{i\frac{3}{2}\alpha t})\,|-+\rangle) \\ &= \frac{1}{\sqrt{2}}(e^{-i(\frac{1}{2}\alpha+\omega)t}\,|++\rangle + e^{i\frac{1}{2}\alpha t}\cos\alpha t\,|+-\rangle - ie^{i\frac{1}{2}\alpha t}\sin\alpha t\,|-+\rangle) \\ &\equiv A\,|++\rangle + B\,|+-\rangle + C\,|-+\rangle) \end{aligned} \tag{41}$$

Tetthetsoperator

$$\begin{aligned} \hat{\rho}(t) = \; & |A|^2\,|++\rangle\langle++| + |B|^2\,|+-\rangle\langle+-| + |C|^2\,|-+\rangle\langle-+| \\ & + AB^*\,|++\rangle\langle+-| + A^*B\,|+-\rangle\langle++| \\ & + AC^*\,|++\rangle\langle-+| + A^*C\,|-+\rangle\langle++| \\ & + BC^*\,|+-\rangle\langle-+| + B^*C\,|-+\rangle\langle+-| \end{aligned} \tag{42}$$

Koeffisienter

$$\begin{aligned} |A|^2 = \frac{1}{2}\,, \quad |B|^2 = \frac{1}{2}\cos^2\alpha t\,, \quad |C|^2 = \frac{1}{2}\sin^2\alpha t \\ AB^* = \frac{1}{2}e^{-i(\alpha+\omega)t}\cos\alpha t\,, \quad A^*B = \frac{1}{2}e^{i(\alpha+\omega)t}\cos\alpha t \\ AC^* = \frac{i}{2}e^{-i(\alpha+\omega)t}\sin\alpha t\,, \quad A^*C = -\frac{i}{2}e^{i(\alpha+\omega)t}\sin\alpha t \\ BC^* = \frac{i}{4}\sin 2\alpha t\,, \quad B^*C = -\frac{i}{4}\sin 2\alpha t \end{aligned} \tag{43}$$

d) Redusert tetthetsoperator for spinn 1

$$\begin{aligned}\hat{\rho}_1(t) &= (|A|^2 + |B|^2)\,|+\rangle\langle+| + |C|^2\,|-\rangle\langle-| + AC^*\,|+\rangle\langle-| + A^*C\,|-\rangle\langle+| \\ &= \frac{1}{2}(1+\cos^2\alpha t)\,|+\rangle\langle+| + \frac{1}{2}\sin^2\alpha t\,|-\rangle\langle-| \\ &\quad + \frac{i}{2}e^{-i(\alpha+\omega)t}\sin\alpha t\,|+\rangle\langle-| - \frac{i}{2}e^{i(\alpha+\omega)t}\sin\alpha\,|-\rangle\langle+| \end{aligned} \tag{44}$$

Benytter

$$|\pm\rangle\langle\pm| = \frac{1}{2}(\mathbb{1} \pm \sigma_z)\,, \quad |\pm\rangle\langle\mp| = \frac{1}{2}(\sigma_x \pm i\sigma_y) \tag{45}$$

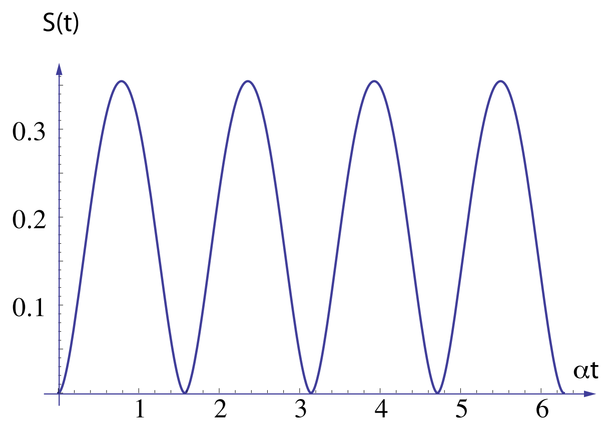
Det gir

$$\hat{\rho}_1(t) = \frac{1}{2}(\mathbb{1} + \cos^2\alpha t\,\sigma_z + \sin[(\alpha+\omega)t]\sin\alpha t\,\sigma_x - \cos[(\alpha+\omega)t]\sin\alpha t\,\sigma_y \tag{46}$$

og

$$\mathbf{r}_1(t) = \sin\alpha t\{\sin[(\alpha+\omega)t]\mathbf{i} - \cos[(\alpha+\omega)t]\mathbf{j}\} + \cos^2\alpha t\,\mathbf{k} \tag{47}$$

Når $\omega >> \alpha$ presseserer vektoren raskt rundt z-aksen, mens vinkelen mellom vektoren og z-aksen gjennomfører en mer langsom periodisk variasjon.



e) Tetthetsoperatoren $\hat{\rho}_1 = \frac{1}{2}(\mathbb{1} + \mathbf{r}_1 \cdot \boldsymbol{\sigma})$ har egenverdier

$$p_\pm = \frac{1}{2}(1 \pm r_1) \tag{48}$$

Sammenfiltringsentropien

$$S = -\left[\frac{1+r_1}{2}\log\frac{1+r_1}{2} + \frac{1-r_1}{2}\log\frac{1-r_1}{2}\right] \tag{49}$$

Tidsavhengighet til r_1,

$$r_1 = [\cos^4\alpha t + \sin^2\alpha t]^{1/2} = [1 - \frac{1}{4}\sin^2 2\alpha t]^{1/2} \tag{50}$$

Lign. (49) og (50) benyttes til å plotte tidsavhengigheten til $S(t)$. Basis-2 logaritme brukes.

Spinn 1 har maksimal blanding når r_1 er minst. Det svarer til størst sammenfiltringsentropi. Den størst mulige verdien for S svarer til $r_1 = 0$, som gir $S_{totmax} = \log 2 = 1.0$. Den maksimale verdi under tidsutviklingen oppnås når $\sin^2 2\alpha t = 1$, som gir $r_1 = \sqrt{3}/2$. Den tilsvarende sammenfiltringsentropien er $S_{max} = 2 - (\sqrt{3}/2)\log[2+\sqrt{3}] = 0.35$.

f) Heisenbergs ligning for det totale spinn er

$$\frac{d}{dt}\hat{\boldsymbol{S}} = \omega\,\mathbf{k} \times \hat{\mathbf{S}} \tag{51}$$

Forventningsverdien er

$$\left\langle \hat{\boldsymbol{S}} \right\rangle = \left\langle \hat{\boldsymbol{S}}_1 \right\rangle + \left\langle \hat{\boldsymbol{S}}_2 \right\rangle = \frac{\hbar}{2}(\mathbf{r}_1 + \mathbf{r}_2) = \frac{\hbar}{2}\mathbf{r} \tag{52}$$

Det gir bevegelsesligning

$$\frac{d\mathbf{r}}{dt} = \omega\,\mathbf{k} \times \mathbf{r} \tag{53}$$

Vektoren $\mathbf{r}$ presseserer om z-aksen med sirkelfrekvens ω. Ved $t = 0$ er vinkelen mellom $\mathbf{r}$ og z-aksen 45^o. Denne vinkelen er konstant under bevegelsen.

Midterm Exam FYS4110, fall semester 2011

Solutions

Problem 1

a) Total spin $\vec{S} = \frac{\hbar}{2}(\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{\sigma}) \equiv \frac{\hbar}{2}(\vec{\Sigma}_A + \vec{\Sigma}_B)$

$$\vec{S}^2 = \frac{\hbar^2}{2}(3\mathbb{1}\otimes\mathbb{1} + \vec{\Sigma}_A\cdot\vec{\Sigma}_B)$$

$$= \frac{\hbar^2}{2}(3\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k)$$

$$\sigma_k\otimes\sigma_k\,|\psi_a\rangle = -|\psi_a\rangle \qquad k = 1,2,3$$

$$\sigma_z\otimes\sigma_z\,|\psi_s\rangle = -|\psi_s\rangle$$

$$\sigma_x\otimes\sigma_x\,|\psi_s\rangle = +|\psi_s\rangle$$

$$\sigma_x\otimes\sigma_x\,|\psi_s\rangle = +|\psi_s\rangle$$

The three cases

I: $\langle\vec{S}^2\rangle_1 = \langle\psi_a|\frac{\hbar^2}{2}(3\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k)|\psi_a\rangle = \underline{0}$

II: $\langle\vec{S}^2\rangle_2 = \langle\psi_s|\frac{\hbar^2}{2}(3\mathbb{1} + \sum_{k=1}^{3}\sigma_k\otimes\sigma_k)|\psi_s\rangle = \underline{2\hbar^2}$

III: $\langle\vec{S}^2\rangle_3 = \frac{1}{2}(\langle\vec{S}^2\rangle_1 + \langle\vec{S}^2\rangle_2) = \underline{\hbar^2}$

$\hat{\rho}_1$ is a spin 0 state, $\hat{\rho}_2$ is a spin 1 state

$\hat{\rho}_3$ is a mixture (incoherent) of spin 0 and spin 1

This means: only $\hat{\rho}_1$ is rotationally invariant.

b) Reduced ~~opera~~ density operators

$$\hat{\rho}_1^A = \mathrm{Tr}_B\left[\tfrac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+| - |+-\rangle\langle-+| - |-+\rangle\langle+-|)\right] \quad (1)$$

cross terms

$$= \tfrac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)_A = \underline{\tfrac{1}{2}\mathbb{1}_A}$$

$\hat{\rho}_2^A = \hat{\rho}_3^A = \hat{\rho}_1^A = \underline{\frac{1}{2}\mathbb{1}_A}$ since the cross terms in (1) do not contribute.

Similarly $\hat{\rho}_1^B = \hat{\rho}_2^B = \hat{\rho}_3^B = \underline{\frac{1}{2}\mathbb{1}_B}$ maximally mixed

$\hat{\rho}_1$ and $\hat{\rho}_2$ are pure states

$\Rightarrow$ entropies $\underline{S_1 = S_2 = 0}$

$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)$ is mixed, with probabilities $p_1 = p_2 = \frac{1}{2}$

$$S_3 = -p_1 \log p_1 - p_2 \log p_2 = \underline{\log 2}$$

Entropies of subsystems

$$S_1^A = S_2^A = S_3^A = \underline{\log 2} = S_1^B = S_2^B = S_3^B$$

Inequality : $S \geq \max\{S_A, S_B\}$

I and II : not satisfied

III : satisfied as equality

Degree of entanglement

I and II are pure states, degree of entanglement measured by the entanglement entropy

$$S_1^A = S_1^B = \underline{\log 2}\,; \quad S_2^A = S_2^B = \underline{\log 2}$$

Case III

$$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) = \frac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+|)$$

$$= \frac{1}{2}(|+\rangle\langle+| \otimes |-\rangle\langle-| + |-\rangle\langle-| \otimes |+\rangle\langle+|)$$

It is a mixture of product states, which means that it is separable (non-entangled)

Degree of entanglement $= \underline{0}$

c) $|\theta\rangle = \cos\theta\,|+\rangle + \sin\theta\,|-\rangle \Rightarrow$

$$S_\theta|\theta\rangle = (\cos\theta\, S_z + \sin\theta\, S_x)|\theta\rangle$$

$$= \frac{\hbar}{2}\left[(\cos\theta\cos\tfrac{\theta}{2} + \sin\theta\sin\tfrac{\theta}{2})|+\rangle + (\sin\theta\cos\tfrac{\theta}{2} - \cos\theta\sin\tfrac{\theta}{2})|-\rangle\right]$$

$$= \frac{\hbar}{2}(\cos\tfrac{\theta}{2}|+\rangle + \sin\tfrac{\theta}{2}|-\rangle) = \underline{(+\tfrac{\hbar}{2})|\theta\rangle} \quad \text{spin up state}$$

$$P_A = \langle \hat{P}(\theta) \rangle_A = Tr_A(\hat{P}(\theta)\hat{\rho}_A)$$
$$= \langle \theta | \tfrac{1}{2} \mathbb{1}_A | \theta \rangle = \underline{\tfrac{1}{2}}$$

This is valid for all three cases I, II, III.

Means that there is equal probability for spin up and spin down in any direction θ.

d)
$$P(\theta,\theta') = Tr(\hat{P}(\theta)\otimes\hat{P}(\theta')\,\hat{\rho})$$
$$= \langle \theta,\theta' | \hat{\rho} | \theta,\theta' \rangle \qquad |\theta,\theta'\rangle = |\theta\rangle \otimes |\theta'\rangle$$

$$\langle +-|\theta,\theta'\rangle = \langle + |\theta\rangle\langle - |\theta'\rangle = \cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}$$
$$\langle -+|\theta,\theta'\rangle = \langle - |\theta\rangle\langle + |\theta'\rangle = \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}$$

implies

case I: $$P_1(\theta,\theta') = \tfrac{1}{2}[\langle\theta\theta'|+-\rangle\langle+-|\theta\theta'\rangle + \langle\theta\theta'|-+\rangle\langle-+|\theta\theta'\rangle - \langle\theta\theta'|+-\rangle\langle-+|\theta\theta'\rangle - \langle\theta\theta'|-+\rangle\langle+-|\theta\theta'\rangle]$$
$$= \tfrac{1}{2}[\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta'}{2} + \sin^2\tfrac{\theta}{2}\cos^2\tfrac{\theta'}{2} - 2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\theta'}{2}]$$
$$= \tfrac{1}{2}(\cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2})^2$$
$$= \underline{\tfrac{1}{2}\sin^2\tfrac{\theta-\theta'}{2}}$$

case II and III:

similar evaluations give

$$\underline{P_2(\theta,\theta') = \tfrac{1}{2}\sin^2\tfrac{\theta+\theta'}{2}} \qquad \underline{P_3(\theta,\theta') = \tfrac{1}{4}(\sin^2\tfrac{\theta-\theta'}{2} + \sin^2\tfrac{\theta+\theta'}{2})}$$

e) Plots of the function $F(\theta, \theta')$ for $\theta' = 0.5\,\theta$ (to the left), 3D plots for variable θ and θ' also included (to the right).

Cases I and II show Bell inequality broken (negative F, colored red in 3D plot).

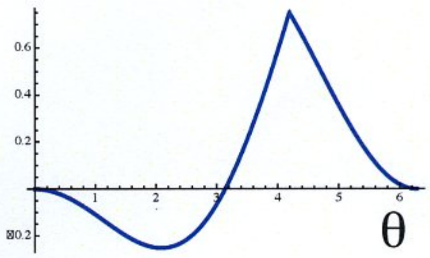
Case III shows no breaking of Bell inequality.

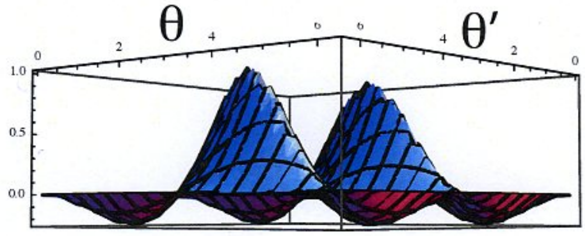
Results consistent with b), I and II being entangled, III being non-entangled.

$\theta' = 0.5\,\theta$

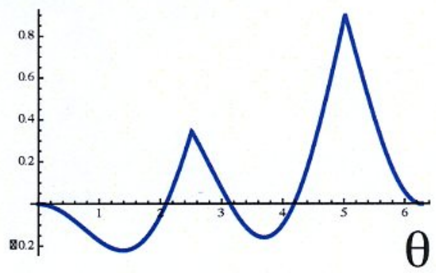
3D plot

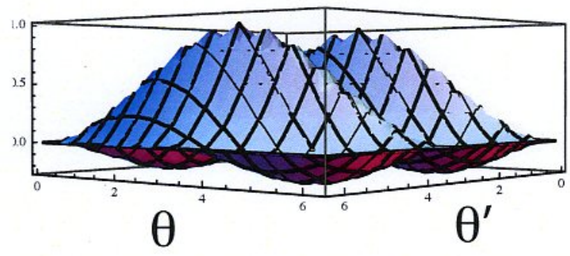
Case I



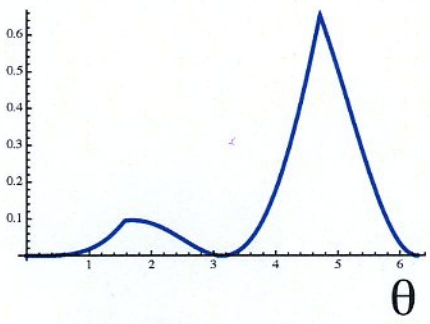


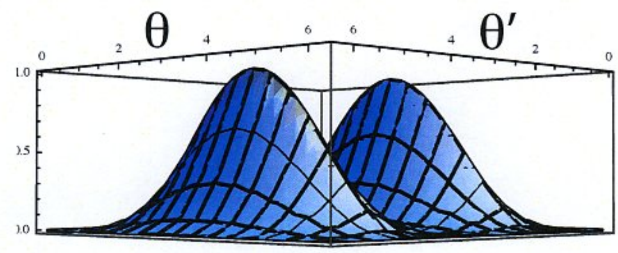
Case II





Case III





f) Experimental quantities

$$P^A_{exp}(\theta) = \frac{n_{++}+n_{+-}}{N} \qquad P^B_{exp}(\theta) = \frac{n_{++}+n_{-+}}{N}$$

$$P^{*}_{exp}(\theta,\theta') = \frac{n_{++}}{N}$$

Problem 2

a) $$\hat{H}_1 |g,n\rangle = -i\hbar\lambda\sqrt{n}\,|e,n-1\rangle$$

$$\hat{H}_1 |e,n-1\rangle = i\hbar\lambda\sqrt{n}\,|g,n\rangle$$

mixes only these two levels

$\Rightarrow$

$$\langle g,n|\hat{H}|g,n\rangle = \hbar(-\tfrac{1}{2}\omega_0 + n\omega)$$

$$\langle e,n-1|\hat{H}|e,n-1\rangle = \hbar(\tfrac{1}{2}\omega_0 + (n-1)\omega)$$

$$\langle g,n|\hat{H}|e,n-1\rangle = i\hbar\lambda\sqrt{n}$$

$$\langle e,n-1|\hat{H}|g,n\rangle = -i\hbar\lambda\sqrt{n}$$

In matrix form

$$H_n = \frac{1}{2}\hbar\begin{pmatrix} -\omega_0 + 2n\omega & -2i\lambda\sqrt{n} \\ -2i\lambda\sqrt{n} & \omega_0 + 2(n-1)\omega \end{pmatrix}$$

$$= \frac{1}{2}\hbar\begin{pmatrix} \omega-\omega_0 & 2i\lambda\sqrt{n} \\ -2i\lambda\sqrt{n} & \omega_0-\omega \end{pmatrix} + \hbar\omega(n\mp\tfrac{1}{2})\mathbb{1}$$

$$\Rightarrow \underline{\Delta = \omega-\omega_0}, \quad \underline{\varepsilon_n = \hbar\omega(n-\tfrac{1}{2})} \quad \underline{\omega_n = 2\lambda\sqrt{n}}$$

$$\hat{H}|g0\rangle = \hat{H}_0|g,0\rangle = -\tfrac{1}{2}\hbar\omega_0|g,0\rangle$$

time evolution $|\psi(0)\rangle = |g,0\rangle \Rightarrow \underline{|\psi(t)\rangle = e^{\frac{i}{2}\omega_0 t}|g,0\rangle}$

b) $\omega = \omega_0 \Rightarrow \Delta = 0$

$$H_n = \frac{1}{2}\hbar\omega_n \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + \varepsilon_n \mathbb{1}$$

$$= \underline{\varepsilon_n \mathbb{1} - \frac{1}{2}\hbar\omega_n \sigma_y}$$

Eigenstates and eigenvalues

$$\sigma_y \phi_n^{\pm} = \mp \phi_n^{\pm} \Rightarrow E_n^{\pm} = \varepsilon_n \pm \frac{1}{2}\hbar\omega_n$$
$$= \underline{\hbar\left[\left(n-\frac{1}{2}\right)\omega \pm \lambda\sqrt{n}\right]}$$

$$\phi_n^{\pm} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \mp\alpha = -i\beta \ , \quad \beta = \mp i\alpha \qquad \underline{\phi_n^{\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp i \end{pmatrix}}$$

General state

$$\psi_n = d_n^+ \phi_n^+ + d_n^- \phi_n^- = \frac{1}{\sqrt{2}}\begin{pmatrix} d_n^+ + d_n^- \\ -i(d_n^+ - d_n^-) \end{pmatrix}$$

$$\Rightarrow c_{n1} = \frac{1}{\sqrt{2}}(d_n^+ + d_n^-) \qquad\qquad d_n^+ = \frac{1}{\sqrt{2}}(c_{n1} + i c_{n2})$$
$$\Rightarrow$$
$$c_{n2} = -\frac{i}{\sqrt{2}}(d_n^+ - d_n^-) \qquad\qquad d_n^- = \frac{1}{\sqrt{2}}(c_{n1} - i c_{n2})$$

Time evolution $d_n^{\pm}(t) = e^{-\frac{i}{\hbar}E_n^{\pm}t}\, d_n^{\pm}(0)$

$$\Rightarrow c_{n1}(t) = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_n^+ t} d_n^+(0) + e^{-\frac{i}{\hbar}E_n^- t} d_n^-(0)\right)$$
$$= \frac{1}{2}\Big(\left(e^{-\frac{i}{\hbar}E_n^+ t} + e^{-\frac{i}{\hbar}E_n^- t}\right) c_{n1}(0)$$
$$+ \frac{i}{2}\left(e^{-\frac{i}{\hbar}E_n^+ t} - e^{-\frac{i}{\hbar}E_n^- t}\right) c_{n2}(0)\Big)$$

$$\Rightarrow c_{n1}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\left(\cos\frac{\omega_n t}{2}\, c_{n1}(0) + \sin\frac{\omega_n t}{2}\, c_{n2}(0)\right)$$

equiv. derivation:
~~Also.~~ $\underline{c_{n2}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\left(\cos\frac{\omega_n t}{2}\, c_{n2}(0) - \sin\frac{\omega_n t}{2}\, c_{n1}(0)\right)}$

c) General state

$$|\psi\rangle = \sum_{ni} c_{ni} |ni\rangle$$

Density operator

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_{ni}\sum_{n'j} c_{ni}\, c^*_{n'j}\, |ni\rangle\langle n'j|$$

matrix elements

$$\underline{\rho_{ni,n'j} = c_{ni}\, c^*_{n'j}}$$

Reduced density operator of the atom

$$\hat{\rho}_{atom} = Tr_{photon}\,\hat{\rho} = \sum_n \underbrace{\langle n|\hat{\rho}|n\rangle}_{\text{photon states}}$$

$$= \sum_n \sum_{n'i} \sum_{n''j} c_{n'i}\, c^*_{n''j}\, \langle n|n'i\rangle\langle n''j|n\rangle$$

$$\langle n|n'1\rangle = |g\rangle\,\delta_{nn'} \equiv |1\rangle\,\delta_{nn'}$$

$$\langle n|n'2\rangle = |e\rangle\,\delta_{n,n'-1} \equiv |2\rangle\,\delta_{n,n'-1}$$

$$\Rightarrow \langle n|n'i\rangle = |i\rangle\,\delta_{(n+i-1),n'}$$

matrix elements

$$\rho_{ij} = \langle i|\hat{\rho}_{atom}|j\rangle = \sum_n \sum_{n'} \sum_{n''} c_{n'i}\, c^*_{n''j}\, \delta_{(n+i-1),n'}\, \delta_{(n+j-1),n''}$$

$$\underline{= \sum_n c_{(n+i-1)i}\, c^*_{(n+j-1)j}}$$

Diagonal elements

$$\rho_{11} = \sum_n |c_{n1}|^2 \quad \text{prob. for atom to be in the gnd. state}$$

$$\rho_{22} = \sum_n |c_{n2}|^2 \quad \text{— " — excited — " —}$$

Initial state $(t=0)$

Case I $\quad \hat{\rho} = |\psi(0)\rangle\langle\psi(0)| = |e\rangle\langle e| \otimes |m-1\rangle\langle m-1|$

$$\hat{\rho}_{atom} = Tr_{photon}\,\hat{\rho} = |e\rangle\langle e|\,\langle m-1|m-1\rangle = |e\rangle\langle e|$$

$$\Rightarrow \underline{\rho_{ij} = \delta_{i2}\delta_{j2}}$$

Case II $\quad \hat{\rho} = |e\rangle\langle e| \otimes |\alpha\rangle\langle\alpha\rangle$

$$\Rightarrow \underline{\rho_{ij} = \delta_{i2}\delta_{j2}}$$

$c_{ni}(0) = \delta_{nm}\,\delta_{i2}$

d) From b):

Case I: $c_{n1}(t) = e^{-\frac{i}{\hbar}\varepsilon_m t}\sin\frac{\omega_m t}{2}\,\delta_{nm}$

$$c_{n2}(t) = e^{-\frac{i}{\hbar}\varepsilon_m t}\cos\frac{\omega_m t}{2}\,\delta_{nm}$$

density matrix

$$\underline{\rho_{11}(t) = \sin^2\frac{\omega_m t}{2}} \qquad \underline{\rho_{22}(t) = \cos^2\frac{\omega_m t}{2}}$$

$$\rho_{12} = \sum_n \sin\frac{\omega_m t}{2}\cos\frac{\omega_m t}{2}\,\delta_{n,m}\,\delta_{n+1,m} \; \underline{=0}$$

$$\rho_{21} = \rho_{12}^* \; \underline{=0}$$

Case II: $c_{n1}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\sin\frac{\omega_n t}{2}\,\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\,e^{-|\alpha|^2/2}$

$$c_{n2}(t) = e^{-\frac{i}{\hbar}\varepsilon_n t}\cos\frac{\omega_n t}{2}\,\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\,e^{-|\alpha|^2/2}$$

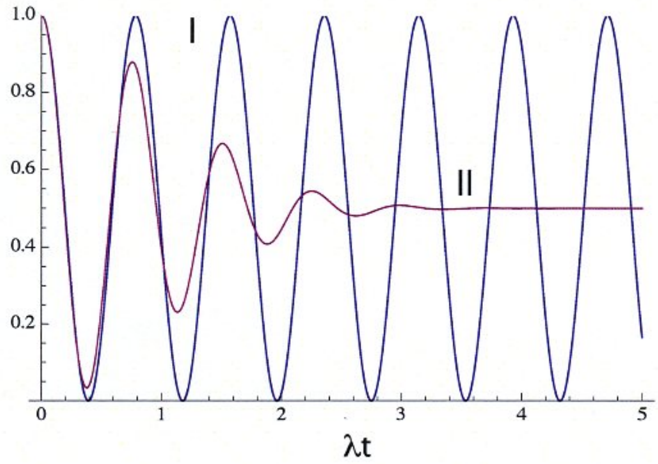
$$\Rightarrow \underline{\rho_{11}(t) = \sum_{n=1}^{\infty}\sin^2\frac{\omega_n t}{2}\,\frac{|\alpha|^{2(n-1)}}{(n-1)!}\,e^{-|\alpha|^2}}$$

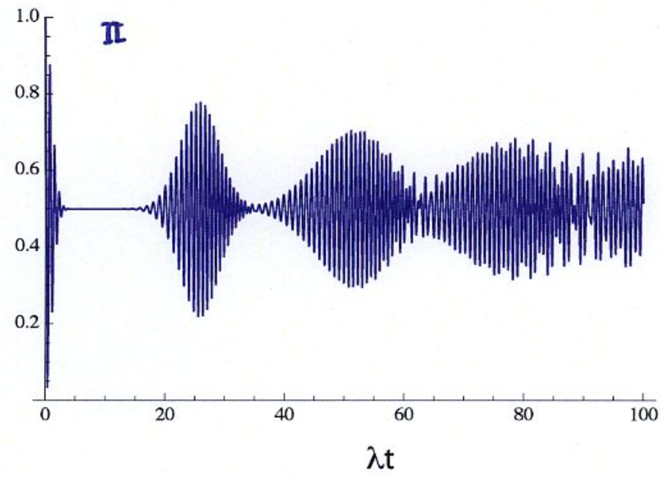
$$\underline{\rho_{22}(t) = \sum_{n=1}^{\infty}\cos^2\frac{\omega_n t}{2}\,\frac{|\alpha|^{2(n-1)}}{(n-1)!}\,e^{-|\alpha|^2}}$$

$$\underline{\rho_{12}(t) = e^{-i\omega t}\sum_{n=1}^{\infty}\sin\frac{\omega_n t}{2}\cos\frac{\omega_{n+1} t}{2}\,\frac{\alpha^{(n-1)}\alpha^{*n}}{\sqrt{(n-1)!\,n!}}\,e^{-|\alpha|^2}}$$

$\rho_{21}(t) = \rho_{12}(t)^* \qquad$ with $\omega_n = 2\lambda\sqrt{n}$

e) Plots of $p_{11}(t)$ for cases I and II,
probability for the atom to be in the excited state.
Case I with the photon number initially defined as $n=4$ shows regular Rabi oscillations between $|e\rangle$ and $|g\rangle$.





Case II, with the e.m. field initially in a coherent state seems first to show damped Rabi oscillations, but the oscillations ~~recovers~~ and show some irregular "quantum beats".

When the atom is excited and de-excited by a classical e.m. field the Rabi oscillations are regular, like in I.

Midterm Exam FYS4110, 2012

Solutions

Problem 1

a) Total spin

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 \quad \Rightarrow \quad S_z = S_{1z} + S_{2z} + S_{3z}$$

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2(\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1)$$

$$= \frac{9}{4}\hbar^2 \mathbb{1} + 2\,(\;-\text{"}-\;)$$

$$\Rightarrow \quad \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_3 + \vec{S}_3 \cdot \vec{S}_1 = \frac{1}{2}\vec{S}^2 - \frac{9}{8}\hbar^2 \mathbb{1}$$

$$\underline{H = \frac{a}{2}\vec{S}^2 + b S_z - \frac{9}{8} a \hbar^2 \mathbb{1}}$$

Spin compositions

spin ½ × spin ½ = spin 0 + spin 1

⇒ spin ½ × (spin ½ × spin ½)

= spin ½ × spin 0 + spin ½ × spin 1

= <u>spin ½ + spin ½ + spin 3/2</u>

b) Lowest energy of the spin ½ subspaces, for $S_z = -\frac{1}{2}\hbar$, is

$$E_0^{1/2} = \frac{a}{2}\frac{3}{4}\hbar^2 - \frac{b}{2}\hbar - \frac{9}{8}a\hbar^2$$

$$\underline{= -\frac{3}{4}a\hbar^2 - \frac{1}{2}b\hbar}$$

Lowest energy for spin 3/2, with $S_z = -\frac{3}{2}\hbar$, is

$$E_0^{3/2} = \frac{a}{2}\frac{15}{4}\hbar^2 - 3\frac{b}{2}\hbar - \frac{9}{8}a\hbar^2$$

$$\underline{= \frac{3}{4}a\hbar^2 - \frac{3}{2}b\hbar}$$

Energy difference

$$E_0^{3/2} - E_0^{1/2} = \frac{3}{2} a\hbar^2 - b\hbar$$

this is positive when $\underline{b < \frac{3}{2} a\hbar}$

This is the condition for the ground state to have spin ½
It is doubly degenerate since the Hamiltonians in the two spin ½ subspaces are identical

c) We examine $|\psi_a\rangle$

$$|\psi_a\rangle = |-\rangle_1 \otimes |\psi_a\rangle_{23}$$

$$|\psi_a\rangle_{23} = \frac{1}{\sqrt{2}}\left(|+-\rangle_{23} - |-+\rangle_{23}\right)$$

This is a spin singlet state (spin 0)
(Is demonstrated by applying $(\vec{S}_2+\vec{S}_3)^2 = 2\vec{S}_2\cdot\vec{S}_3 + \frac{3}{2}\hbar^2\mathbb{1}$ to the state $|\psi_a\rangle_{23}$)

1: The composition of any spin ½ state with a spin 0 state is a spin ½ state.

2: z-component $S_z|\psi_a\rangle = \frac{\hbar}{2}(-1+1-1)|\psi_a\rangle = \underline{-\frac{\hbar}{2}|\psi_a\rangle}$

⇒ The state lies in the subspace of the ground state.

The states $|\psi_b\rangle$ and $|\psi_c\rangle$:
They are derived from $|\psi_a\rangle$ by cyclic permutations of the three spins : 123 → 231 → 312
The total spin $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$ is invariant under permutations ⇒ the three states have the same spin quantum numbers ⇒ they all lie in the subspace of the degenerate ground state.

d) Partition $1+(23)$ for $|\psi_a\rangle$:

$$\rho_a = |\psi_a\rangle\langle\psi_a| = (|-\rangle\langle-|)_1 \otimes (|\psi_a\rangle\langle\psi_a|)_{23}$$

$\Rightarrow$ $\rho_a = \rho_{a1} \otimes \rho_{a23}$ product state

There is no correlation $\Rightarrow$ no entanglement with respect to this partition

Partition $2+(31)$:

$$\rho_{a2} = Tr_{13}\,\rho_a = Tr_1\,\rho_{a1}\; Tr_3\,\rho_{a23} \qquad Tr_1\,\rho_{a1} = 1$$

$$= \frac{1}{2} Tr_3 \left(|+-\rangle\langle+-| + |-+\rangle\langle-+| - |+-\rangle\langle-+| - |-+\rangle\langle+-| \right)_{23}$$

$$= \frac{1}{2} \left(|+\rangle\langle+| + |-\rangle\langle-| \right)_2$$

$$= \frac{1}{2} \mathbb{1}_2$$

Entropy: $S_{a2} = \log 2$

This is the maximal entropy, since the spin space of particle 2 is of dimension 2.

It is the entanglement entropy of the composite system $1+(23)$

Partition $3+(12)$

The density operator is symmetric with respect to the permutation $1 \leftrightarrow 2$ $\Rightarrow$ $\rho_{a3} = \frac{1}{2}\mathbb{1}_3$

$\Rightarrow$ $S_{a3} = \log 2$: maximally mixed

Since $|\psi_b\rangle$ and $|\psi_c\rangle$ are derived from $|\psi_a\rangle$ by permutations, the conclusions are the same up to permutation of spin labels:

$|\psi_b\rangle$ $\quad 123 \rightarrow 231$

$|\psi_c\rangle$ $\quad 123 \rightarrow 312$

e)

$$\langle \Phi_I | \Phi_{II} \rangle = \frac{1}{3}\left(1 + e^{4\pi i/3} + e^{-4\pi i/3}\right)$$

$$= \frac{1}{3}\left(1 + e^{-2\pi i/3} + e^{2\pi i/3}\right)$$

$$e^{\pm 2\pi i/3} = \cos(2\pi/3) \pm i \sin(2\pi/3)$$

$$= -\frac{1}{2} \pm i \frac{1}{2}\sqrt{3}$$

$$\Rightarrow e^{2\pi i/3} + e^{-2\pi i/3} = -1$$

$$\Rightarrow \langle \Phi_I | \psi_{II} \rangle = \frac{1}{3}(1-1) = \underline{0} \qquad \text{orthogonal}$$

If $|\psi_I\rangle$ belongs to the subspace:

$$|\Phi_I\rangle = \alpha|\psi_a\rangle + \beta|\psi_b\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|-+-\rangle - (\alpha-\beta)|--+\rangle - \beta|+--\rangle\right)$$

$$\Rightarrow \frac{\alpha}{\sqrt{2}} = \frac{1}{\sqrt{3}}\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) \qquad (1)$$

$$\frac{\beta}{\sqrt{2}} = -\frac{1}{\sqrt{3}} \qquad (2)$$

$$\frac{\alpha-\beta}{\sqrt{2}} = -\frac{1}{\sqrt{3}}\left(\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) \qquad (3)$$

Consistency check:

$$(1)-(2): \quad \frac{\alpha-\beta}{\sqrt{2}} = \frac{1}{\sqrt{3}}\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) + \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right)$$

the same as (3)

$$\Rightarrow |\Phi_I\rangle = \sqrt{\frac{2}{3}}\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right)|\psi_a\rangle - \sqrt{\frac{2}{3}}|\psi_b\rangle$$

With $|\psi_{II}\rangle$:

$$e^{\pm 2\pi i/3} \rightarrow e^{\mp 2\pi i/3}$$

$$\Rightarrow \underline{|\psi_{II}\rangle = \sqrt{\frac{2}{3}}\left(-\frac{1}{2} - i\frac{1}{2}\sqrt{3}\right)|\psi_a\rangle - \sqrt{\frac{2}{3}}|\psi_b\rangle}$$

Both belong to the subspace

f) Density operator

$$\rho_I = \frac{1}{3}\big(|{+}{-}{-}\rangle\langle{+}{-}{-}| + |{-}{+}{-}\rangle\langle{-}{+}{-}| + |{-}{-}{+}\rangle\langle{-}{-}{+}|$$
$$+ e^{-2\pi i/3} |{+}{-}{-}\rangle\langle{-}{+}{-}| + e^{2\pi i/3} |{+}{-}{-}\rangle\langle{-}{-}{+}|$$
$$+ e^{2\pi i/3} |{-}{+}{-}\rangle\langle{+}{-}{-}| + e^{-2\pi i/3} |{-}{-}{+}\rangle\langle{+}{-}{-}|$$
$$+ e^{-2\pi i/3} |{-}{+}{-}\rangle\langle{-}{-}{+}| + e^{2\pi i/3} |{-}{-}{+}\rangle\langle{-}{+}{-}| \big)$$

Reduced density operators

$$\rho_{I1} = \frac{1}{3}\left(|+\rangle\langle+| + |-\rangle\langle-| + |-\rangle\langle-| \right)_1$$
$$= \frac{1}{3}(|+\rangle\langle+|)_1 + \frac{2}{3}(|-\rangle\langle-|)_1$$

Entropy $S_{I1} = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = \log 3 - \frac{2}{3}\log 2$

$$\rho_{I2} = \frac{1}{3}\left(|-\rangle\langle-| + |+\rangle\langle+| + |-\rangle\langle-| \right)_2$$
$$= \frac{1}{3}(|+\rangle\langle+|)_2 + \frac{2}{3}(|-\rangle\langle-|)_2$$

$$\rho_{I3} = \frac{1}{3}(|+\rangle\langle+|)_3 + \frac{2}{3}(|-\rangle\langle-|)_3$$

$$\Rightarrow \quad S_{I1} = S_{I2} = S_{I3} = \underline{\log 3 - \frac{2}{3}\log 2}$$

The results are precisely the same for $|\Psi_{II}\rangle$

Comparison with the average entanglement entropy of $|\psi_a\rangle$ ($|\psi_b\rangle$ and $|\psi_c\rangle$):

$$\bar{S}_a = \frac{2}{3}\log 2$$

Difference $S_I - \bar{S}_a = \log 3 - \frac{4}{3}\log 2$

$\log_2$: $S_I - \bar{S}_a = \log_2 3 - \frac{4}{3} = 0.25 > 0$

g) Measurement of S_{1z} in the state $|\psi_I\rangle$

If measured result is $S_{1z} = +\frac{\hbar}{2}$, the spin of particle 1 is projected into the state $|+\rangle_1$

$\Rightarrow$ The full state is changed to:

$$|\psi_I\rangle \to |+--\rangle = |+\rangle_1 \otimes |-\rangle_2 \otimes |-\rangle_3$$

This is a pure product state, with no entanglement

If measured result is $S_{1z} = -\frac{\hbar}{2}$, the spin of particle 1 is projected into the state $|-\rangle_1$.

$\Rightarrow$ The full state is changed to

$$|\psi_I\rangle \to \frac{1}{\sqrt{2}} |-\rangle_1 \otimes \left(e^{2\pi i/3} |+-\rangle_{23} + e^{-2\pi i/3} |-+\rangle_{23} \right)$$

for the (23) subsystem

$$\rho_{23} = \frac{1}{2} \left(|+-\rangle\langle+-| + |-+\rangle\langle-+| \right)_{23}$$

and the reduced density operators are

$$\rho_2 = \frac{1}{2} \left(|+\rangle\langle+| + |-\rangle\langle-| \right)_2 = \frac{1}{2} \mathbb{1}_2$$

similarly

$$\rho_3 = \frac{1}{2} \mathbb{1}_3$$

The entanglement entropy of subsystem 23 then is $\underline{S = \log 2}$

Problem 2

$$\vec{A} = -\tfrac{1}{2}\vec{r}\times\vec{B} = -\tfrac{B}{2}\vec{r}\times\vec{k}$$

$$\Rightarrow A_x = -\tfrac{1}{2}By, \quad A_y = \tfrac{1}{2}Bx$$

Introduce $\vec{\pi} = \vec{p} - e\vec{A} \Rightarrow \pi_x = p_x + \frac{1}{2}eBy; \; \pi_y = p_y - \frac{1}{2}eBx$

$$H = \frac{1}{2m}\vec{\pi}^2 = \frac{1}{2m}(\pi_x^2 + \pi_y^2)$$

a) $L = (\vec{r}\times\vec{p})_z = xp_y - yp_x$

$$\begin{aligned}
[L,\pi_x] &= [xp_y - yp_x, p_x + \tfrac{1}{2}eBy] \\
&= [x,p_x]p_y + \tfrac{1}{2}eBx[p_y,y] \\
&= i\hbar(p_y - \tfrac{1}{2}eBx) \\
&= i\hbar\pi_y
\end{aligned}$$

$$\begin{aligned}
[L,\pi_y] &= [xp_y - yp_x, p_y - \tfrac{1}{2}eBx] \\
&= -[y,p_y]p_x + \tfrac{1}{2}eBy[p_x,x] \\
&= -i\hbar(p_x + \tfrac{1}{2}eBy) \\
&= -i\hbar\pi_x
\end{aligned}$$

$$\begin{aligned}
[L,H] &= \frac{1}{2m}[L,\pi_x^2 + \pi_y^2] \\
&= \frac{1}{2m}\left([L,\pi_x]\pi_x + \pi_x[L,\pi_x] + [L,\pi_y]\pi_y + \pi_y[L,\pi_y]\right) \\
&= \frac{i\hbar}{2m}(\pi_y\pi_x + \pi_x\pi_y - \pi_x\pi_y - \pi_y\pi_x) = \underline{0}
\end{aligned}$$

L commutes with $H \Rightarrow L$ is a constant of motion b)

$$X = x + \frac{1}{m\omega}\pi_y \qquad m\omega = eB$$

$$= x + \frac{1}{eB}(p_y - \tfrac{1}{2}eBx)$$

$$= \tfrac{1}{2}x + \frac{1}{eB}p_y$$

$$Y = y - \frac{1}{m\omega}\pi_x$$

$$= y - \frac{1}{eB}(p_x + \tfrac{1}{2}eBy)$$

$$= \tfrac{1}{2}y - \frac{1}{eB}p_x$$

$$\Rightarrow [X,Y] = [\tfrac{1}{2}x, -\tfrac{1}{eB}p_x] + [\tfrac{1}{eB}p_y, \tfrac{1}{2}y] = -\frac{i\hbar}{eB} = -i\, l_B^2$$

$$[a,a^+] = \frac{1}{2l_B^2}([X,+iY] + [-iY,X])$$

$$= +\frac{i}{l_B^2}[X,Y] = \underline{1}$$

Similarly

$$\eta_x = \frac{1}{eB}\pi_y = -\tfrac{1}{2}x + \frac{1}{eB}p_y$$

$$\eta_y = -\frac{1}{eB}\pi_x = -\tfrac{1}{2}y - \frac{1}{eB}p_x$$

$$[\eta_x, \eta_y] = \frac{1}{2eB}\{[x,p_x] - [p_y,y]\} = i\, l_B^2$$

$$[b,b^+] = \frac{1}{2l_B^2}(-2i)[\eta_x,\eta_y] = \underline{1}$$

$$[X,\eta_x] = [Y,\eta_y] = 0$$

$$[X,\eta_y] = [\tfrac{1}{2}x + \tfrac{1}{eB}p_y, -\tfrac{1}{2}y - \tfrac{1}{eB}p_x] = 0$$

$$[Y,\eta_x] = [\tfrac{1}{2}y - \tfrac{1}{eB}p_x, -\tfrac{1}{2}x + \tfrac{1}{eB}p_y] = 0$$

$$\Rightarrow [a,b] = [a^+,b] = [a,b^+] = 0$$

Commut. relations as for two independent harm. oscillators

c) $$H = \frac{(eB)^2}{2m}(\eta_x^2+\eta_y^2)$$

$$= \frac{(eB)^2}{2m}\frac{l_B^2}{2}\left((b+b^\dagger)^2-(b-b^\dagger)^2\right)$$

$$= \frac{1}{2}\hbar\omega(b^\dagger b+bb^\dagger) = \underline{\hbar\omega(b^\dagger b+\tfrac{1}{2})}$$

Constant energy splitting $\hbar\omega$, as for harmonic oscillator

Independent of $a, a^\dagger$, implies all energy eigenstates reached by a and $a^\dagger$ have the same energy

Lowest energy states $b|\psi\rangle = 0 \Rightarrow E_0 = \frac{1}{2}\hbar\omega$

Define $a|0\rangle = 0$ and $b|0\rangle = 0$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \Rightarrow b|n\rangle = 0$$

all have the same energy E_0

Angular momentum

$$x = X-\eta_x,\ y = Y-\eta_y$$

$$p_x = -\frac{eB}{2}(Y+\eta_y),\ p_y = \frac{eB}{2}(X+\eta_x)$$

$$\Rightarrow L = \left[(X-\eta_x)(X+\eta_x)+(Y-\eta_y)(Y+\eta_y)\right]\frac{eB}{2}$$

$$= \frac{eB}{2}(X^2+Y^2-\eta_x^2-\eta_y^2)$$

$$= \frac{eB}{2}\frac{l_B^2}{2}\left((a+a^\dagger)^2-(a-a^\dagger)^2-(b+b^\dagger)^2+(b-b^\dagger)^2\right)$$

$$= \frac{1}{2}\hbar(aa^\dagger+a^\dagger a-bb^\dagger-b^\dagger b)$$

$$= \underline{\hbar(a^\dagger a-b^\dagger b)}$$

$$\Rightarrow L|n\rangle = \hbar a^\dagger a|n\rangle = n\hbar|n\rangle$$

angular momentum $l_n = n\hbar$

d) $|z,-z\rangle_a = N(z)\left(|z\rangle \otimes |-z\rangle - |-z\rangle \otimes |z\rangle\right)$

$$(a_1+a_2)|z,-z\rangle_a = (z-z)|z,-z\rangle_a = 0 \quad \text{eigenvalue } 0$$

$$a_1 a_2 |z,-z\rangle_a = z(-z)|z,-z\rangle_a = \underline{-z^2 |z,-z_a\rangle}$$

Normalization

$$\langle z,-z|z,-z\rangle_a = 1$$

$$\Rightarrow \quad |N(z)|^2 \big(\langle z|z\rangle\langle -z|-z\rangle + \langle -z|-z\rangle\langle z|z\rangle$$
$$- \langle z|-z\rangle\langle -z|z\rangle - \langle -z|z\rangle\langle z|-z\rangle\big)$$

$$= 2|N(z)|^2 \left(1 - |\langle z|-z\rangle|^2\right) \overset{!}{=} 1$$

$$\langle z|-z\rangle = e^{-\frac{1}{2}(|z|^2+|-z|^2) + z^*(-z)} = e^{-2|z|^2}$$

$$\Rightarrow \quad 2|N(z)|^2\left(1-e^{-4|z|^2}\right)$$

$$\Rightarrow \quad \underline{N(z) = \frac{1}{\sqrt{2(1-e^{-4|z|^2})}}}$$

Density operators

$$\rho = |z,-z\rangle_a \langle z,-z|_a = |N(z)|^2$$
$$* \big(|z\rangle\langle z| \otimes |-z\rangle\langle -z| + |-z\rangle\langle -z| \otimes |z\rangle\langle z|$$
$$- |z\rangle\langle -z| \otimes |-z\rangle\langle z| - |-z\rangle\langle z| \otimes |z\rangle\langle -z|\big)$$

Reduced density operators

$$\rho_1 = |N(z)|^2 \big(|z\rangle\langle z| + |-z\rangle\langle -z|$$
$$- |z\rangle\langle -z| \langle z|-z\rangle - |-z\rangle\langle +z| \langle -z|z\rangle\big)_1$$

$$\underline{= \frac{1}{2(1-e^{-4|z|^2})}\left(|z\rangle\langle z| + |-z\rangle\langle -z| - e^{-2|z|^2}\left(|z\rangle\langle -z| + |-z\rangle\langle z|\right)\right)}$$

same expression for ρ_2

e) Density matrix in the coherent state representation

$$\rho_1(z,z') = \langle z|\hat{\rho}_1|z'\rangle$$

$$= |N(Z)|^2 \big(\langle z|Z\rangle\langle Z|z'\rangle + \langle z|-Z\rangle\langle -Z|z'\rangle$$

$$- e^{-2|Z|^2} (\langle z|Z\rangle\langle -Z|z'\rangle + \langle z|-Z\rangle\langle Z|z'\rangle)\big)$$

$$\langle z|Z\rangle = e^{-\frac{1}{2}(|z|^2+|Z|^2)+z^*Z} \Rightarrow$$

$$\rho_1(z,z') = |N(Z)|^2 e^{-|Z|^2} e^{-\frac{1}{2}(|z|^2+|z'|^2)}$$

$$\times \left((e^{z^*Z+Z^*z'} + e^{-(z^*Z+Z^*z')}) - e^{-2|Z|^2} (e^{z^*Z-Z^*z'} + e^{-z^*Z+Z^*z'}) \right)$$

$$= \frac{e^{-|Z|^2}}{1-e^{-4|Z|^2}} e^{-\frac{1}{2}(|z|^2+|z'|^2)} \big(\cosh(z^*Z+Z^*z')$$

$$- e^{-2|Z|^2} \cosh(z^*Z-Z^*z') \big)$$

One-particle density

$$\rho(z) = 2\rho_1(z,z)$$

$$= 2 \frac{e^{-(|Z|^2+|z|^2)}}{1-e^{-4|Z|^2}} \left(\cosh(2\mathrm{Re}(z^*Z)) - e^{-2|Z|^2} \cos(2\,\mathrm{Im}(z^*Z))\right)$$

Assume Z real

$$\rho(z) = 2 \frac{e^{-(Z^2+|z|^2)}}{1-e^{-4Z^2}} \left(\cosh(2Z\,\mathrm{Re}\,z) - e^{-2Z^2} \cos(2Z\,\mathrm{Im}\,z)\right)$$

Plots for $Z = 2, 1, 0.1$

$Z=2$ two particles far apart, two gaussians

$Z=1$ the two parts begin to merge

$Z=0.1$ the two parts on the top of each others, not a fully gaussian form, flattened on the top, due to Pauli exclusion

f) $\hat{\rho}_1 |z\rangle = |N(z)|^2 \{ |z\rangle (1 - e^{-4|z|^2}) + |-z\rangle (e^{-2|z|^2} - e^{-2|z|^2}) \}$

$= \frac{1}{2} |z\rangle$

$\hat{\rho}_1 |-z\rangle = |N(z)|^2 \{ |-z\rangle (1 - e^{-4|z|^2}) + |z\rangle (e^{-2|z|^2} - e^{-2|z|^2}) \}$

$= \frac{1}{2} |-z\rangle$

$\Rightarrow \hat{\rho}_1 = \frac{1}{2} \hat{P}$ $\hat{P}$ projection on subspace spanned by $|z\rangle$ and $|-z\rangle$

(note $\hat{\rho}_1 |\psi\rangle = 0$ for any state orthogonal to both $|z\rangle$ and $|-z\rangle$)

$\Rightarrow \hat{\rho}_1 = \frac{1}{2} (|1\rangle\langle 1| + |2\rangle\langle 2|)$

with $|1\rangle$ and $|2\rangle$ as orthonormalized states in this subspace

Entropy $S_1 = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = \log 2$

$\Rightarrow$ entanglement entropy of two-particle system.

Entanglement is due to antisymmetrization, Fermi-Dirac statistics

g) N particles in the lowest angular momentum states

Antisymmetric state

$$|\psi\rangle = \mathcal{N}_1 \left(|0,1,2, \cdots (N-1)\rangle - |1,0,2, \ldots, (N-1)\rangle + \cdots \right)$$

N! permutations, sign change for odd number of interchange of pair of particle indices.

Normalization : $\langle\psi|\psi\rangle = |\mathcal{N}_1|^2 \cdot N!$ $\qquad \mathcal{N}_1 = \frac{1}{\sqrt{N!}}$

Density operator

$$\rho = |\psi\rangle\langle\psi| = |\mathcal{N}_1|^2 \big(|0,1 \ldots (N-1)\rangle\langle 0,1 \cdots (N-1)|$$
$$+ |1,0, \ldots, (N-1)\rangle\langle 1,0, \ldots (N-1)| + \cdots \big)$$

N! terms, all with weight +1

Particle 1 (first position) all angular momenta appear with the same weight

Reduced density operator

$$\hat{\rho}_1 = \mathrm{Tr}_{2,3,N-1}\, \hat{\rho} = N_2 \left(|0\rangle\langle 0| + |1\rangle\langle 1| + \cdots |N-1\rangle\langle N-1| \right)$$

Normalization $\mathrm{Tr}\,\hat{\rho}_1 = 1 \Rightarrow |N_2|^2 N = 1$ $\quad N_2 = \frac{1}{\sqrt{N}}$

$$\Rightarrow \quad \underline{\hat{\rho}_1 = \frac{1}{N} \sum_{n=0}^{N-1} |n\rangle\langle n|}$$

One-particle density

$$\rho_1(z) = N \rho_1(z,z) = \sum_{n=0}^{N-1} |\langle z|n\rangle|^2$$
$$= \underline{\sum_{n=0}^{N-1} \frac{|z|^{2n}}{n!} e^{-|z|^2}}$$

Plot of $\rho(z)$ for $N=10$:

Almost constant density $\rho(z) \simeq 1$ for $|z|^2 \lesssim \sqrt{10}$

Increase in the density prohibited by Pauli exclusion principle the lowest angular momenta occupy the area with lowest $|z|^2$. This means that the density of the inner part cannot be increased by adding particles

h) Plot of $\rho(z)$ for $N=2$

Looks precisely the same as the two-particle coherent state for $Z=0.01$.

Limit $Z \to 0$:

Two-particle coherent state, Z real

One particle density:

$$\rho(z) = \frac{2e^{-Z^2}}{1-e^{-4Z^2}} e^{-|z|^2}\left(\cosh(2Z\,\mathrm{Re}z) - e^{-2Z^2}\cos(2Z\,\mathrm{Im}z)\right)$$

$Z \to 0$, expand in Z^2 to first order

$e^{-Z^2} \simeq 1 - Z^2, \quad 1-e^{-4Z^2} \simeq 4Z^2$

$\cosh(2Z\,\mathrm{Re}z) \simeq 1 + 2Z^2(\mathrm{Re}z)^2; \quad \cos(2Z\,\mathrm{Im}z) \simeq 1 - 2Z^2(\mathrm{Im}z)^2$

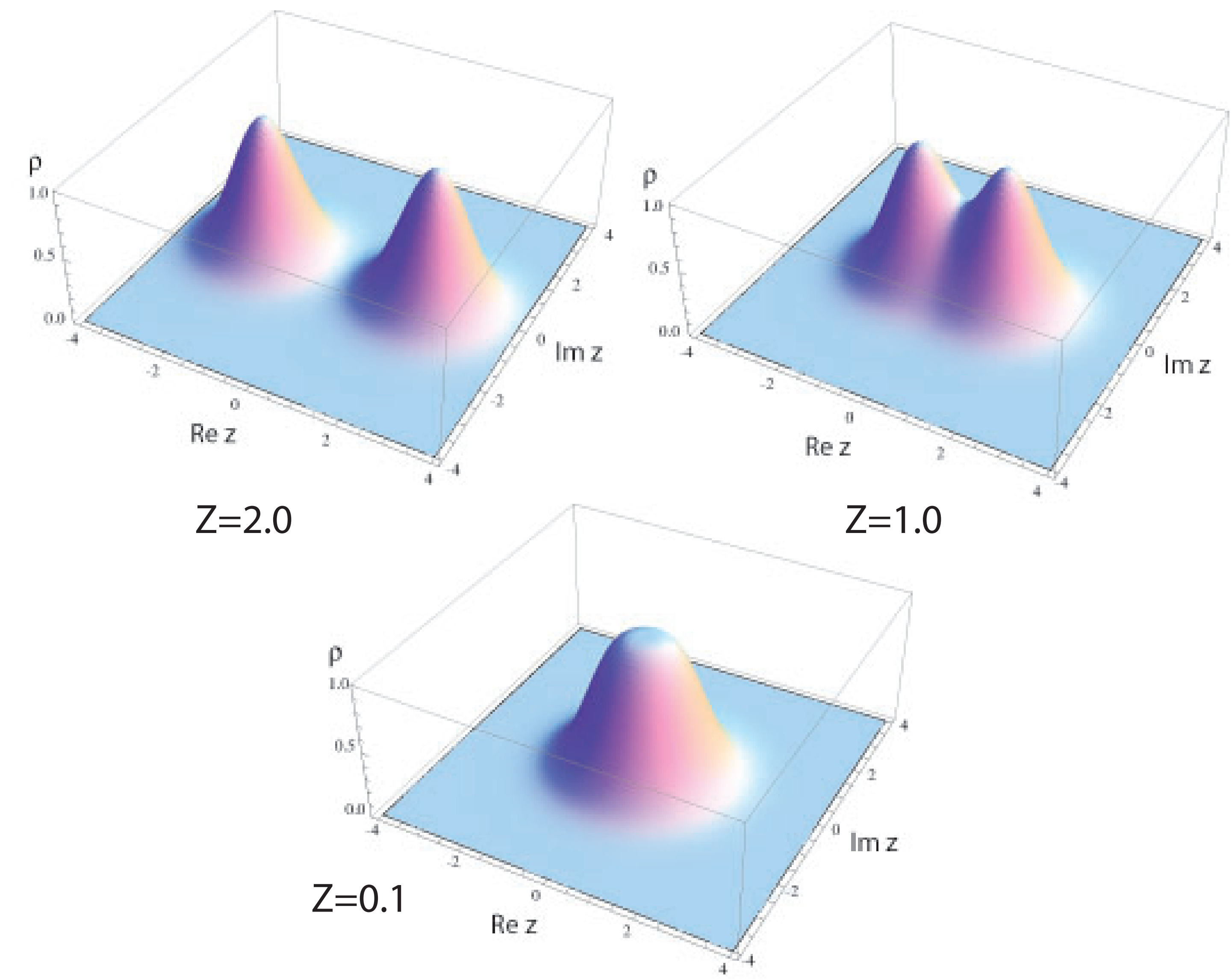
$\cosh(2Z\,\mathrm{Re}z) - e^{-2Z^2}\cos(2Z\,\mathrm{Im}z) \simeq 2Z^2(1+|z|^2)$

$$\rho(z) \simeq \frac{2(1-Z^2)}{4Z^2} e^{-|z|^2} 2Z^2(1+|z|^2) \simeq e^{-|z|^2}(1+|z|^2) + \mathcal{O}(Z^2)$$

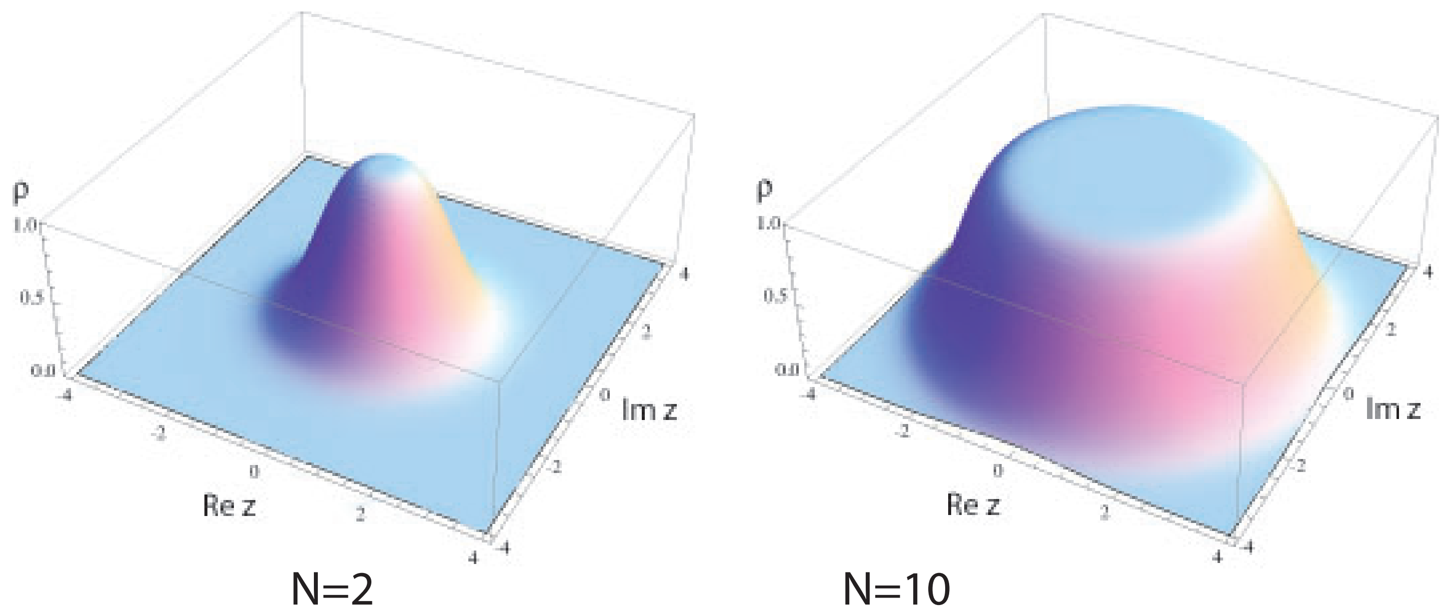
$$\lim_{Z\to 0} \rho(z) = e^{-|z|^2}(1+|z|^2) = \underline{\sum_{n=0}^{1} \frac{|z|^{2n}}{n!} e^{-|z|^2}}$$

same as when ang. mom $l=0$ and $l=1$ are occupied

Antisymmetrized coherent states



Angular momentum states



Midtterminekssamen FYS4110, høsten 2013

Løsninger

Oppgave 1

a) Benytter produktregelen for Paulimatriser:

$$\hat{\rho}^2 = \frac{1}{16}\Big[(1+\vec{a}^2+\vec{b}^2+\sum_{ij} c_{ij}^2)\,\mathbb{1}\otimes\mathbb{1}$$
$$+ 2\sum_i (a_i + \sum_j c_{ij} b_j)\,\sigma_i\otimes\mathbb{1}$$
$$+ 2\sum_j (b_j + \sum_i a_i c_{ij})\,\mathbb{1}\otimes\sigma_j$$
$$+ \sum_{ij} (2c_{ij} + 2a_i b_j - \sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} c_{kl} c_{mn})\,\sigma_i\otimes\sigma_j \Big]$$

Reduserte tetthetsmatriser,

benytter $\mathrm{Tr}\,\sigma_i = 0 \quad i=1,2,3$: , $\mathrm{Tr}\,\mathbb{1} = 2$ for hvert delsystem

$$\hat{\rho}_A = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma}), \quad \hat{\rho}_B = \frac{1}{2}(\mathbb{1} + \vec{b}\cdot\vec{\sigma})$$

$$\hat{\rho}_A^2 = \frac{1}{4}((1+\vec{a}^2)\mathbb{1} + 2\vec{a}\cdot\vec{\sigma}), \quad \hat{\rho}_B^2 = \frac{1}{4}((1+\vec{b}^2)\mathbb{1} + 2\vec{b}\cdot\vec{\sigma})$$

b) Spektralutvikling av $\hat{\rho}$

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \quad \text{med } 0 \le p_k \le 1, \ \sum_k p_k = 1$$

og $\langle\psi_k|\psi_l\rangle = \delta_{kl}$

$$\Rightarrow \hat{\rho}^2 = \sum_k p_k^2 |\psi_k\rangle\langle\psi_k|$$

med $p_k^2 \le p_k$

$$\Rightarrow \mathrm{Tr}\,\hat{\rho}^2 \le \mathrm{Tr}\,\hat{\rho}$$

Likhet $p_k^2 = p_k \Rightarrow p_k = 1$ eller 0,

kan bare oppnås med $p_k = 1$ for én k-verdi

$\Rightarrow \hat{\rho} = |\psi\rangle\langle\psi|$, dvs ren tilstand

Betingelse på koeffisienter

$$\mathrm{Tr}\hat{\rho}^2 = \frac{1}{4}\left(1 + \vec{a}^2 + \vec{b}^2 + \sum_{ij} c_{ij}^2\right) \leq 1$$

$$\Leftrightarrow \quad \underline{\vec{a}^2 + \vec{b}^2 + \sum_{ij} c_{ij}^2 \leq 3}$$

Tilsvarende $\mathrm{Tr}_A \hat{\rho}_A^2 \leq 1 \Rightarrow \underline{\vec{a}^2 \leq 1}$

$\mathrm{Tr}_B \hat{\rho}_B^2 \leq 1 \Rightarrow \underline{\vec{b}^2 \leq 1}$

c)

Anta $\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$ tensorprodukttilstand

med $\hat{\rho}_A = \frac{1}{2}(\mathbb{1} + \vec{a}\cdot\vec{\sigma})$; $\hat{\rho}_B = \frac{1}{2}(\mathbb{1} + \vec{b}\cdot\vec{\sigma})$

$$\Rightarrow \quad \hat{\rho} = \frac{1}{4}(\mathbb{1} + \vec{a}\cdot\vec{\sigma}) \otimes (\mathbb{1} + \vec{b}\cdot\vec{\sigma})$$

$$= \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\vec{b}\cdot\vec{\sigma} + \sum_{ij} a_i b_j \sigma_i \otimes \sigma_j\right)$$

$$\Rightarrow \quad \underline{c_{ij} = a_i b_j}$$

Anta $\hat{\rho}$ ren og maksimalt sammenfiltret,

dvs $\hat{\rho}_A$ og $\hat{\rho}_B$ er maksimalt blandet:

$$\hat{\rho}^2 = \hat{\rho}, \quad \hat{\rho}_A = \frac{1}{2}\mathbb{1}_A, \quad \hat{\rho}_B = \frac{1}{2}\mathbb{1}_B$$

$$\Rightarrow \quad \vec{a} = \vec{b} = 0$$

$$\hat{\rho}^2 = \frac{1}{16}\left[\left(1 + \sum_{ij} c_{ij}^2\right)\mathbb{1}\otimes\mathbb{1} + 2\sum_{ij}\left(c_{ij} - \frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} c_{kl} c_{mn}\right)\sigma_i \otimes \sigma_j\right.$$

$$\hat{\rho}^2 = \hat{\rho} \quad \Rightarrow$$

$$\underline{\sum_{ij} c_{ij}^2 = 3} \quad \& \quad \underline{\frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} c_{kl} c_{mn} = -c_{ij}}$$

d) Oversettelse fra bra-ket-notasjon

$$|\pm\rangle\langle\pm| = \frac{1}{2}(\mathbb{1} \pm \sigma_z)$$

$$|\pm\rangle\langle\mp| = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

$$\Rightarrow \quad |++\rangle\langle++| + |--\rangle\langle--| = \frac{1}{2}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$$

$$|++\rangle\langle--| + |--\rangle\langle++| = \frac{1}{2}(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)$$

$$\Rightarrow \quad \underline{\hat{\rho}_{B1} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)}$$

$$\underline{\hat{\rho}_{B2} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)}$$

B1 & B2 : $\vec{a} = \vec{b} = 0 \quad c_{ij} = c_i\delta_{ij}$

B1 : $c_x = +1, \; c_y = -1, \; c_z = +1$

B2 : $c_x = -1, \; c_y = +1, \; c_z = +1$

$c_{ij} = c_i\delta_{ij} \Rightarrow$

$$\sum_{ij} c_{ij}^2 = \sum_i c_i^2 = \underline{3} \qquad \text{for B1 \& B2}$$

$$\frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} c_{kl} c_{mn} = \frac{1}{2}\sum_{km} \varepsilon_{kmi}\varepsilon_{kmj} c_k c_m$$

$$= \frac{1}{2}\delta_{ij}\sum_{km} \varepsilon_{kmi}^2 c_k c_m$$

$i=j=1: \quad = \frac{1}{2}(\varepsilon_{231}^2 + \varepsilon_{321}^2)c_2c_3 = c_2c_3 = \mp 1$

$i=j=2: \quad = \frac{1}{2}(\varepsilon_{312}^2 + \varepsilon_{132}^2)c_1c_3 = c_1c_3 = \pm 1$

$i=j=3: \quad = \frac{1}{2}(\varepsilon_{123}^2 + \varepsilon_{213}^2)c_1c_2 = c_1c_2 = -1$

likhet med $-c_{ij} = -c_i\delta_{ij}$:

$i=j=1: \quad = -c_1 = \mp 1$

$i=j=2: \quad = -c_2 = \pm 1$

$i=j=3 \quad = -c_3 = -1$

dvs: betingelse $\frac{1}{2}\sum_{klmn} \varepsilon_{kmi}\varepsilon_{lnj} c_{kl} c_{mn} = -c_{ij}$ <u>er oppfylt</u>

e) $\hat{\rho}_1(t) = \cos^2\omega t\, \hat{\rho}_{B1} + \sin^2\omega t\, \hat{\rho}_{B2}$

$+ \cos\omega t \sin\omega t\, (|B1\rangle\langle B2| + |B2\rangle\langle B1|)$

$|B1\rangle\langle B2| + |B2\rangle\langle B1| = \frac{1}{2}(|++\rangle\langle ++| - |--\rangle\langle --|)$

$= \frac{1}{2}(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1})$

$\Rightarrow \hat{\rho}_1(t) = \frac{1}{4}\big(\mathbb{1}\otimes\mathbb{1} + \sin(2\omega t)(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}) + \cos(2\omega t)(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)$
$\underline{+ \sigma_z\otimes\sigma_z\big)}$

$\left.\begin{array}{l}\hat{\rho}_A(t) = \frac{1}{2}(\mathbb{1} + \sin(2\omega t)\,\sigma_z) \\ \hat{\rho}_B(t) = \text{— " —}\end{array}\right\}$ egenverdier $\underline{p_\pm = \frac{1}{2}(1 \pm \sin(2\omega t))}$

Sammenfiltringsentropi $S_e(t) = \underline{-p_+\log p_+ - p_-\log p_-}$

f) $\hat{\rho}_2(t) = \cos^2\omega t\, \hat{\rho}_{B1} + \sin^2\omega t\, \hat{\rho}_{B2}$

$\hat{\rho}_2(t)|B1\rangle = \cos^2\omega t\, |B1\rangle$

$\hat{\rho}_2(t)|B2\rangle = \sin^2\omega t\, |B2\rangle$

Entropi $S(t) = \underline{-\cos^2\omega t \log(\cos^2\omega t) - \sin^2\omega t \log(\sin^2\omega t)}$

$\hat{\rho}_A = \rho_B = \frac{1}{2}\mathbb{1} \Rightarrow \underline{S_A = S_B = \log 2}$

g) $\omega t = \frac{\pi}{4}$

$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|B1\rangle + |B2\rangle) = |++\rangle = |+\rangle\otimes|+\rangle$

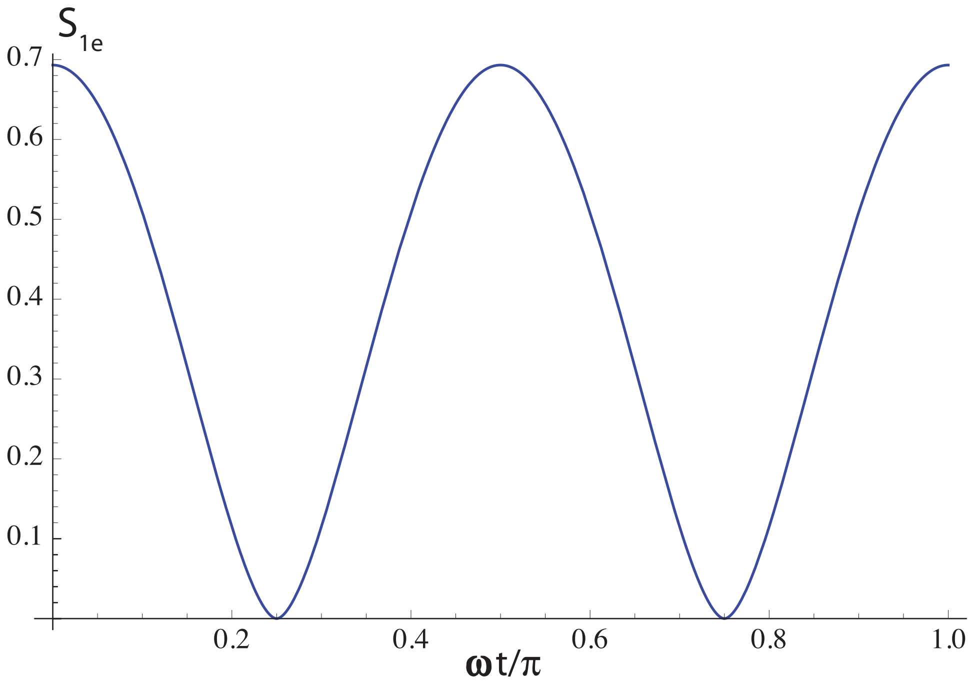
ren produkttilstand $\Rightarrow$ separabel $\underline{\hat{\rho}_1 = |+\rangle\langle +| \otimes |+\rangle\langle +|}$

$\hat{\rho}_2 = \frac{1}{2}(\hat{\rho}_{B1} + \hat{\rho}_{B2}) = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\sigma_z)$

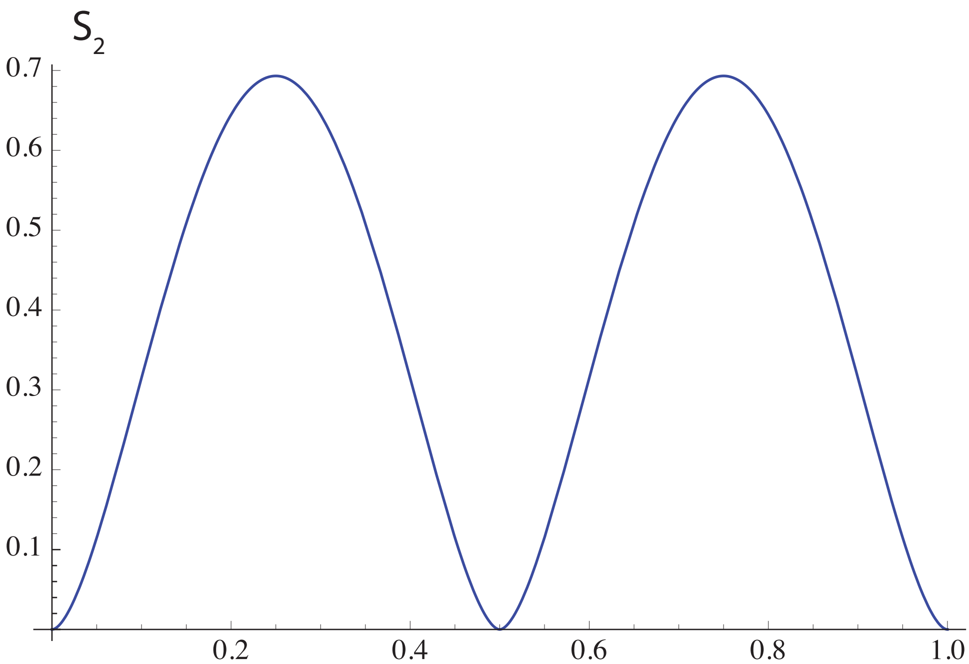
$= \underline{\frac{1}{2}\left\{\left[\frac{1}{2}(\mathbb{1}+\sigma_z)\right]\otimes\left[\frac{1}{2}(\mathbb{1}+\sigma_z)\right] + \left[\frac{1}{2}(\mathbb{1}-\sigma_z)\right]\otimes\left[\frac{1}{2}(\mathbb{1}-\sigma_z)\right]\right\}}$

sum av to produkttilstander $\Rightarrow$ separabel

Oppgave 1 e)
Sammenfiltringsentropi
(målt i naturlig logaritme)



Oppgave 1 f)
Von Neumann-entropi



Oppgave 2

a) $\hat{H}|g,1\rangle = (\frac{1}{2}\hbar\omega - i\gamma\hbar)|g,1\rangle + \frac{1}{2}\hbar\lambda|e,0\rangle$

$\hat{H}|e,0\rangle = \frac{1}{2}\hbar\omega|e,0\rangle + \frac{1}{2}\hbar\lambda|g,1\rangle$

($\hat{H}|g,0\rangle = -\frac{1}{2}\hbar\omega|g,0\rangle$ frakoblet de andre)

I 2-dim. underrom,

$$\hat{H} = \begin{pmatrix} \frac{1}{2}\hbar\omega & \frac{1}{2}\hbar\lambda \\ \frac{1}{2}\hbar\lambda & \frac{1}{2}\hbar(\omega - 2i\gamma) \end{pmatrix} = \underline{\frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix}}$$

b) Tidsutviklingsoperatoren kan skrives som

$$\hat{U}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}\, e^{-i\vec{\Omega}\cdot\vec{\sigma}t}$$

med $\vec{\Omega} = \frac{1}{2}(\lambda\vec{i} + i\gamma\vec{k})$

$$e^{-i\vec{\Omega}\cdot\vec{\sigma}t} = 1 - i\vec{\Omega}\cdot\vec{\sigma}t + \frac{1}{2!}(-i\vec{\Omega}\cdot\vec{\sigma}t)^2 + \dots + \frac{1}{n!}(-i\vec{\Omega}\cdot\vec{\sigma}t)^n + \dots$$

Utnytter $(\vec{\Omega}\cdot\vec{\sigma})^2 = \vec{\Omega}^2 \equiv \Omega^2$

$\Rightarrow \quad (\vec{\Omega}\cdot\vec{\sigma})^3 = \Omega^2\,\vec{\Omega}\cdot\vec{\sigma}$ etc

Skiller mellom like og odde potenser

$$e^{-i\vec{\Omega}\cdot\vec{\sigma}t} = 1 - \frac{1}{2}\Omega^2t^2 + \frac{1}{4!}\Omega^4t^4 + \dots$$
$$- i\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma}\left(\Omega t - \frac{1}{3!}\Omega^3t^3 + \dots\right)$$
$$= \cos(\Omega t) - i\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma}\sin(\Omega t)$$

$$\Rightarrow \quad \underline{\hat{U}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}\left(\cos\Omega t - i\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma}\sin\Omega t\right)}$$

korrekt form med $\vec{\Omega} = \frac{1}{2}(\lambda\vec{i} + i\gamma\vec{k})$

$\Rightarrow \quad \vec{\Omega}^2 = \frac{1}{4}(\lambda^2 - \gamma^2)$ reell og positiv når $\lambda > \gamma$

$$\Rightarrow \quad \underline{\Omega = \frac{1}{2}\sqrt{\lambda^2 - \gamma^2}}$$

c) På matriseform

$$\psi(t) = \hat{U}(t)\,\psi(0)$$

$$= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t & -i\frac{\lambda}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t & \cos\Omega t - \frac{\gamma}{2\Omega}\sin\Omega t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix} \cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t \end{pmatrix}$$

$$\Rightarrow \quad |\psi(t)\rangle = \underline{e^{-\frac{1}{2}(i\omega+\gamma)t}\left[\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)|e,0\rangle\right.}$$
$$\underline{\left. - i\frac{\lambda}{2\Omega}\sin\Omega t\,|g,1\rangle\right]}$$

d) $\mathrm{Tr}\,\hat{\rho}(t) = \langle\psi(t)|\psi(t)\rangle$

$$= e^{-\gamma t}\left(\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)^2 + \frac{\lambda^2}{4\Omega^2}\sin^2\Omega t\right)$$

$$= \underline{e^{-\gamma t}\left(\frac{\lambda^2}{4\Omega^2} - \frac{\gamma^2}{4\Omega^2}\cos 2\Omega t + \frac{\gamma}{2\Omega}\sin 2\Omega t\right)}$$

$\mathrm{Tr}\,\hat{\rho}_{cav} = 1 \quad \Rightarrow$

$$f(t) = \underline{1 - \mathrm{Tr}\,\hat{\rho}(t)} = 1 - \langle\psi(t)|\psi(t)\rangle$$

Ved utsendelse av fotonet gjennom kavitetsveggen vil systemet ende opp i tilstand $|g,0\rangle$. Tillegget til $\hat{\rho}$ sørger for at det skjer slik at den samlete sannsynlighet for at atomet er i en av tilstandene $|e\rangle$ og $|g\rangle$ er konstant, lik 1.

e) Besetningssannsynligheter for atomet

$$p_e(t) = \langle e,0|\hat{\rho}_{tot}(t)|e,0\rangle$$
$$= \langle e,0|\hat{\rho}(t)|e,0\rangle$$
$$= |\langle \psi(t)|e,0\rangle|^2$$
$$= e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma}{2\Omega}\sin\Omega t\right)^2$$
$$= \underline{e^{-\gamma t}\left(\frac{\lambda^2}{8\Omega^2} + \frac{\lambda^2-2\gamma^2}{8\Omega^2}\cos 2\Omega t + \frac{\gamma}{2\Omega}\sin 2\Omega t\right)}$$

$$p_g(t) = \underline{1 - p_e(t)}$$

Sannsynlighet for et foton i kaviteten

$$p_f(t) = \langle g,1|\hat{\rho}(t)|g,1\rangle$$
$$= |\langle \psi(t)|g,1\rangle|^2$$
$$= \underline{\frac{\lambda^2}{8\Omega^2}e^{-\gamma t}(1-\cos 2\Omega t)}$$

f) $$\hat{\rho}_{cav}(t) = |\psi(t)\rangle\langle\psi(t)| + f(t)|g,0\rangle\langle g,0|$$
$$= \langle\psi(t)|\psi(t)\rangle\, |\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| + \text{— " —}$$
$$= \underline{(1-f(t))|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)| + f(t)|g,0\rangle\langle g,0|}$$

hvor $\langle\tilde{\psi}(t)|\tilde{\psi}(t)\rangle = 1$

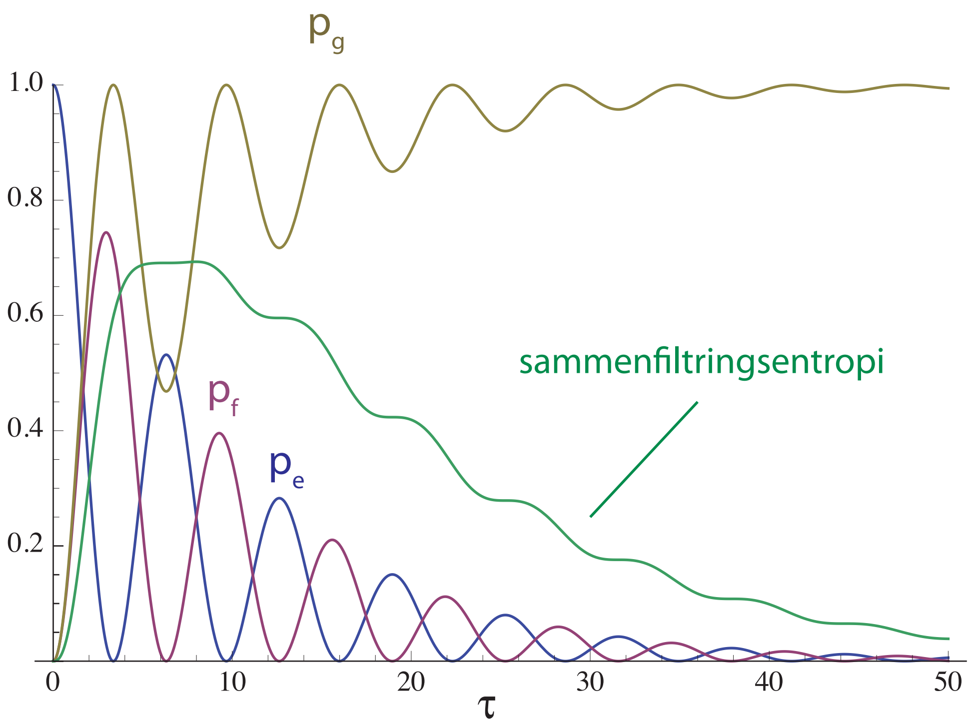
Dette er en spektralutvikling av $\hat{\rho}_{tot}$ siden $\langle\tilde{\psi}|g,0\rangle = 0$

Egenverdiene er $f(t)$ og $1-f(t)$.

Entropi $S = \underline{-f\log f - (1-f)\log(1-f)}$

er lik sammenfiltringsentropien til det sammensatte systemet.

Oppgave 2 e) og f)
Besetningssannsynligheter og
sammenfiltringsentropi



FYS4110 Midterm Exam 2014

Solutions

Problem 1 Spin splitting in positronium

a) $\langle ij | \vec{\Sigma}_e \cdot \vec{\Sigma}_p | kl \rangle$

$$= \sum_{m,n} \langle ij | \vec{\sigma}_e \otimes \mathbb{1}_p | mn \rangle \cdot \langle mn | \mathbb{1}_e \otimes \vec{\sigma}_p | kl \rangle$$

$$= \sum_{m,n} (\langle i | \vec{\sigma}_e | m \rangle \delta_{jn}) \cdot (\delta_{mk} \langle n | \vec{\sigma}_p | l \rangle)$$

$$= \underline{\langle i | \vec{\sigma}_e | k \rangle \cdot \langle j | \vec{\sigma}_p | l \rangle}$$

b) Matrix elements

$$\vec{\sigma} = \sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k} \Rightarrow$$

$$\langle + | \vec{\sigma} | + \rangle = \vec{k} \; ; \; \langle - | \vec{\sigma} | - \rangle = -\vec{k}$$

$$\langle + | \vec{\sigma} | - \rangle = \vec{i} - i\vec{j} \; , \; \langle - | \vec{\sigma} | + \rangle = \vec{i} + i\vec{j}$$

$$\Rightarrow \langle ++ | \vec{\Sigma}_e \cdot \vec{\Sigma}_p | ++ \rangle = \vec{k} \cdot \vec{k} = 1$$

$$\langle ++ | \text{— " —} | +- \rangle = \vec{k} \cdot (\vec{i} - i\vec{j}) = 0$$

$$\langle ++ | \text{— " —} | -+ \rangle = \text{— " —} = 0$$

$$\langle ++ | \text{— " —} | -- \rangle = (\vec{i} - i\vec{j})^2 = 0$$

$$\langle +- | \text{— " —} | +- \rangle = -\vec{k} \cdot \vec{k} = -1$$

$$\langle +- | \text{— " —} | -+ \rangle = (\vec{i} - i\vec{j}) \cdot (\vec{i} + i\vec{j}) = 2$$

$$\langle +- | \text{— " —} | -- \rangle = (\vec{i} - i\vec{j}) \cdot (-\vec{k}) = 0$$

$$\langle -+ | \text{— " —} | -+ \rangle = (-\vec{k}) \cdot \vec{k} = -1$$

$$\langle -+ | \text{— " —} | -- \rangle = (-\vec{k}) \cdot (\vec{i} - i\vec{j}) = 0$$

$$\langle -- | \text{— " —} | -- \rangle = (-\vec{k})^2 = 1$$

other terms determined by hermiticity of $\vec{\Sigma}_e \cdot \vec{\Sigma}_p$

Matrix representation

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) From b) follows

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |0,0\rangle = \frac{1}{\sqrt{2}} \left(\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |+-\rangle - \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |-+\rangle \right)$$

$$= -\frac{3}{4}\hbar^2 |0,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,1\rangle = \hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |++\rangle = \frac{\hbar^2}{4} |1,1\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,0\rangle = \frac{\hbar^2}{4} |1,0\rangle$$

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p |1,-1\rangle = \frac{\hbar^2}{4} |1,-1\rangle$$

In the spin basis,

$$\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p = \frac{\hbar^2}{4} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Total spin $\hat{\vec{S}}^2 = (\hat{\vec{S}}_e + \hat{\vec{S}}_p)^2 = \hat{\vec{S}}_e^2 + \hat{\vec{S}}_p^2 + 2\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$

$$= \frac{\hbar^2}{4}\left[(\vec{\sigma}_e \otimes \mathbb{1}_p)^2 + (\mathbb{1}_e \otimes \vec{\sigma}_p)^2 \right] + 2\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

$$= \frac{3}{2}\hbar^2 \mathbb{1} + 2\hat{\vec{S}}_e \cdot \hat{\vec{S}}_p$$

$\Rightarrow$ in spin basis

$$\hat{\vec{S}}^2 = 2\hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad S_z = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\vec{S}^2 = s(s+1)\hbar^2 \Rightarrow$ $s = 0$ for $|0,0\rangle$ singlet

$s = 1$ for $|1,m\rangle$ $m = 0, \pm 1$ triplet

d) Need to find the matrix elements of $(S_e)_z - (S_p)_z \equiv D$

$$D|1,1\rangle = D|1,-1\rangle = 0$$

$$D|0,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle - (-2)|-+\rangle\right) = \hbar|1,0\rangle$$

$$D|1,0\rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\left(2|+-\rangle + (-2)|-+\rangle\right) = \hbar|0,0\rangle$$

mixes only $|0,0\rangle$ and $|1,0\rangle$

Hamiltonian in the spin basis

$$H = \begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & 0 & \lambda\hbar^2 & 0 \\ 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 & 0 \\ \lambda\hbar^2 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa & 0 \\ 0 & 0 & 0 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}$$

e) · $|1,1\rangle$ and $|1,-1\rangle$ are eigenvectors with eigenvalues $\underline{E = E_0 + \frac{1}{4}\hbar^2\kappa}$ (indep. of λ)

Eigenvalue problem for the remaining two states

$$\begin{pmatrix} E_0 - \frac{3}{4}\hbar^2\kappa & \lambda\hbar^2 \\ \lambda\hbar^2 & E_0 + \frac{1}{4}\hbar^2\kappa \end{pmatrix}\begin{pmatrix}\gamma \\ \delta\end{pmatrix} = E\begin{pmatrix}\gamma \\ \delta\end{pmatrix}$$

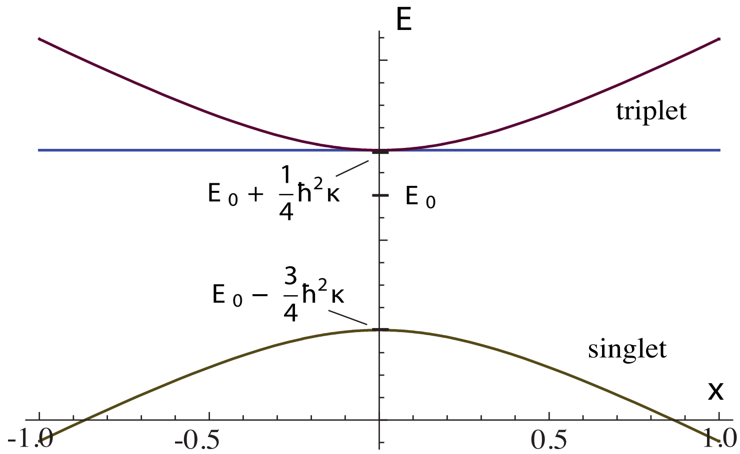
write this as $\left(E_0 - \frac{1}{4}\hbar^2\kappa\right)\mathbb{1} + \frac{1}{2}\hbar^2\kappa\begin{pmatrix} -1 & 2x \\ 2x & 1\end{pmatrix}$ $\qquad x = \lambda/\kappa$

$$\Rightarrow \begin{pmatrix} -1 & 2x \\ 2x & 1\end{pmatrix}\begin{pmatrix}\gamma \\ \delta\end{pmatrix} = \mu\begin{pmatrix}\gamma \\ \delta\end{pmatrix} \quad \text{with} \quad E = E_0 - \frac{1}{4}\hbar^2\kappa + \frac{1}{2}\hbar^2\kappa\mu$$

eigenvalues $\begin{vmatrix} -1-\mu & 2x \\ 2x & 1-\mu\end{vmatrix} = 0 \Rightarrow \mu^2 = 4x^2 + 1$

$$E_\pm = E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\kappa\sqrt{4x^2+1}$$

$$= \underline{E_0 - \frac{1}{4}\hbar^2\kappa \pm \frac{1}{2}\hbar^2\sqrt{\kappa^2 + 4\lambda^2}}$$



f) $\hat{\rho}_a = |a\rangle\langle a| = |\alpha|^2 |+-\rangle\langle+-| + |\beta|^2 |-+\rangle\langle-+|$

$\qquad + \alpha\beta^* |+-\rangle\langle-+| + \alpha^*\beta |-+\rangle\langle+-|$

$\hat{\rho}_b = |b\rangle\langle b| = |\beta|^2 |+-\rangle\langle+-| + |\alpha|^2 |-+\rangle\langle-+|$

$\qquad - \alpha\beta^* |+-\rangle\langle-+| - \alpha^*\beta |-+\rangle\langle+-|$

Reduced density operators

$\hat{\rho}_{ae} = Tr_p\, \hat{\rho}_a = |\alpha|^2 |+\rangle\langle+| + |\beta|^2 |-\rangle\langle-|$

$\hat{\rho}_{ap} = Tr_e\, \hat{\rho}_a = |\alpha|^2 |-\rangle\langle-| + |\beta|^2 |+\rangle\langle+|$

$\hat{\rho}_{be} = Tr_p\, \hat{\rho}_B = |\beta|^2 |+\rangle\langle+| + |\alpha|^2 |-\rangle\langle-|$

$\hat{\rho}_{bp} = Tr_e\, \hat{\rho}_B = |\beta|^2 |-\rangle\langle-| + |\alpha|^2 |+\rangle\langle+|$

g. Entropy

$S_{ae} = S_{ap} = S_{be} = S_{bp} = -\left(|\alpha|^2 \log|\alpha|^2 + |\beta|^2 \log|\beta|^2\right)$

$$= -\left(|\alpha|^2 \log|\alpha|^2 + (1-|\alpha|^2)\log(1-|\alpha|^2)\right)$$

g) Eigenstates

$$|a\rangle = \gamma|0,0\rangle + \delta|1,0\rangle = \alpha|+-\rangle + \beta|-+\rangle$$

$$\Rightarrow \quad \alpha = \frac{\gamma+\delta}{\sqrt{2}} \;, \quad \beta = \frac{\gamma-\delta}{\sqrt{2}}$$

γ, δ determined by eigenvalue eq. in e):

$$-\gamma + 2x\delta = \mu\gamma \quad \Rightarrow \quad \delta = \frac{\mu+1}{2x}\gamma$$

$\mu = \pm\sqrt{4x^2+1}$; choose $\mu = -\sqrt{4x^2+1}$ (+ gives $|b\rangle$)

gives $\delta \to 0$ for $x \to 0$

Note γ, δ real.

Normalization : $\gamma^2 + \delta^2 = \left(1 + \left(\frac{\mu+1}{2x}\right)^2\right)\gamma^2 = 1$

$$\Rightarrow \quad \gamma^2 = \frac{4x^2}{4x^2+(\mu+1)^2}$$

$$\alpha^2 = \frac{1}{2}\left(1 + \frac{\mu+1}{2x}\right)^2\gamma^2 = \frac{1}{2}\,\frac{(2x+\mu+1)^2}{4x^2+(\mu+1)^2}$$

α

$$(2x+\mu+1)^2 = 4x^2+1+4x+\mu^2+2(2x+1)\mu$$
$$= 2\left(\mu^2 + 2x(\mu+1)+\mu\right) = 2(\mu+1)(\mu+2x)$$

$$4x^2+(\mu+1)^2 = 4x^2+1+\mu^2+2\mu = 2(\mu^2+\mu) = 2\mu(\mu+1)$$

$$\Rightarrow \quad \alpha^2 = \frac{1}{2}\,\frac{2(\mu+2x)(\mu+1)}{2\mu(\mu+1)} = \underline{\frac{1}{2}\left(1 + \frac{2x}{\sqrt{4x^2+1}}\right)}$$

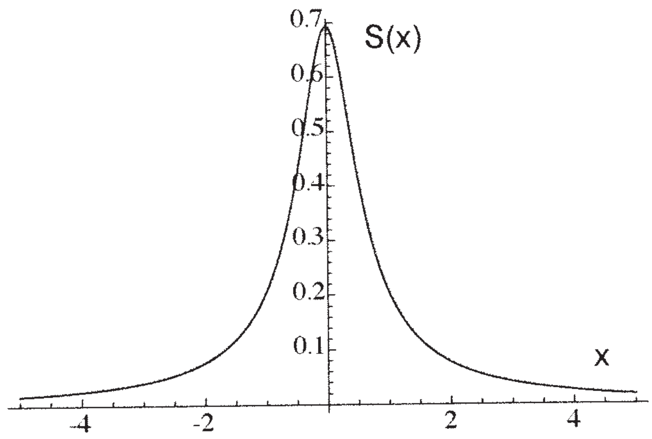
$$\beta^2 = 1-\alpha^2 = \frac{1}{2}\left(1 - \frac{2x}{\sqrt{4x^2+1}}\right)$$

Entropy of reduced density matrices

$$\underline{S(x) = -\left[\alpha(x)^2 \log\alpha(x)^2 + \beta(x)^2 \log\beta(x)^2\right]}$$

For $x = 0$: $\hat{\rho}_{ae} = \hat{\rho}_{be} = \frac{1}{2}\mathbb{1}_e$, $\hat{\rho}_{ap} = \hat{\rho}_{bp} = \frac{1}{2}\mathbb{1}_p$

maximal entanglement $S(0) = \log 2$



Entanglement of states |a> and |b> as functions of $x = \lambda/\kappa$

Problem 2, Spin-coherent states

a) Eigenvalue equation

$$\hat{J}_- |\psi\rangle = \lambda |\psi\rangle , \quad |\psi\rangle = \sum_m c_m |j,m\rangle$$

Since $m \leq j$, there must be a maximum value, $m \leq m_{max}$ in the expansion. Application of $\hat{J}_-$ reduces $m \Rightarrow m_{max} \to m_{max}-1$.
Repeated application $\Rightarrow \hat{J}_-^{2j+1} |\psi\rangle = 0 = \lambda^{2j+1} |\psi\rangle$
This implies $\lambda = 0$, which is satisfied only for $|\psi\rangle = |j,-j\rangle$
Similar argument for $\hat{J}_+$ gives eigenvalue $= 0$ also for this operator. This is satisfied only for $|\psi\rangle = |j,j\rangle$.

b) $\hat{\vec{J}}^2 = j(j+1)\hbar^2 \Rightarrow (\Delta\vec{J})^2 = j(j+1)\hbar^2 - \langle \hat{\vec{J}} \rangle^2$

Implies: min. value for $(\Delta\vec{J})^2 \Leftrightarrow$ max. value for $\langle \hat{\vec{J}} \rangle^2$.

For general state, define unit vector $\vec{n}$ by

$$\langle \vec{J} \rangle = J\vec{n} , \quad J^2 = \langle \vec{J} \rangle^2$$

This gives $\langle \vec{J} \rangle^2 = \langle J_{\vec{n}} \rangle^2 \quad \hat{J}_{\vec{n}} = \vec{n}\cdot\hat{\vec{J}}$

Rotational invariance $\Rightarrow$
all directions equivalent, may choose z-axis with $\vec{k} = \vec{n}$

For $\vec{n} = \vec{k}$:

$$\langle \vec{J} \rangle^2 = \langle J_z \rangle^2 , \quad \langle J_x \rangle = \langle J_y \rangle = 0$$

$$\Rightarrow \langle \vec{J} \rangle^2 \leq j(j+1)\hbar^2 \quad \text{since} \quad -j\hbar \leq \langle J_z \rangle \leq j\hbar$$

Inequality valid for all directions $\vec{n}$.

For $\vec{n} = \vec{k}$:

max. value for $\langle \vec{J} \rangle^2$ for $\langle \hat{J}_z \rangle^2 = j^2 \hbar^2$,

which is the case for the states $|j,-j\rangle$ and $|j,j\rangle$

For general $\vec{n}$ this corresponds to

$$\underline{\hat{J}_{\vec{n}} |j,\vec{n}\rangle = j\hbar |j,\vec{n}\rangle}$$

with $|j,\vec{n}\rangle$ denoting the eigenstate of $\hat{J}_{\vec{n}}$ with maximal eigenvalue. Note: all min. uncertainty states are then included, since $j \to -j$ is equivalent to $\vec{n} \to -\vec{n}$.

Minimum uncertainty value

$$(\Delta \vec{J})^2 = j(j+1)\hbar^2 - j^2\hbar^2 = \underline{j\hbar^2}$$

c) Spin $j = 1/2$

$\vec{J} = \frac{\hbar}{2}\vec{\sigma}$, use standard representation of Pauli matrices

$$\Rightarrow \vec{\sigma} = \sigma_x \vec{i} + \sigma_y \vec{j} + \sigma_z \vec{k} = \begin{pmatrix} \vec{k} & \vec{i} - i\vec{j} \\ \vec{i} + i\vec{j} & -\vec{k} \end{pmatrix}$$

General spin state $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\quad |\alpha|^2 + |\beta|^2 = 1$

$$\langle \vec{J} \rangle = \frac{\hbar}{2} \psi^\dagger \vec{\sigma} \psi = \frac{\hbar}{2} (\alpha^* \beta^*) \begin{pmatrix} \vec{k} & \vec{i} - i\vec{j} \\ \vec{i} + i\vec{j} & -\vec{k} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$= \frac{\hbar}{2} \left((\alpha^*\beta + \alpha\beta^*)\vec{i} + i(\alpha\beta^* - \alpha^*\beta)\vec{j} + (|\alpha|^2 - |\beta|^2)\vec{k} \right)$$

$$\Rightarrow \langle \vec{J} \rangle^2 = \frac{\hbar^2}{4} \left((\alpha^*\beta + \alpha\beta^*)^2 + (\alpha\beta^* - \alpha^*\beta)^2 + (|\alpha|^2 - |\beta|^2)^2 \right)$$

$$= \frac{\hbar^2}{4} (|\alpha|^2 + |\beta|^2)^2 = \frac{\hbar^2}{4} = \underline{j^2\hbar^2} \quad \text{for } j = \tfrac{1}{2}$$

$\langle \vec{J} \rangle^2$ maximal $\Rightarrow (\Delta \vec{J})^2$ minimal, valid for all ψ.

d) Coherent state, $j = 1/2$

$$\vec{\sigma}\cdot\vec{n}\,|z\rangle = |z\rangle \Rightarrow \sum_{m'} \langle m|\vec{\sigma}\cdot\vec{n}|m'\rangle\langle m'|z\rangle = \langle m|z\rangle$$

Matrix form

$$\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$ with $|z\rangle = \alpha\,|+\tfrac{1}{2}\rangle + \beta\,|-\tfrac{1}{2}\rangle$, $n_x = \vec{n}\cdot\vec{i}$ etc

$$\Rightarrow (n_z - 1)\,\alpha + (n_x - i n_y)\,\beta = 0$$

$$\Rightarrow (1-\cos\theta)\,\alpha = e^{-i\varphi}\sin\theta\,\beta$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{\sin\theta}{1-\cos\theta}\,e^{-i\varphi} = \cot\tfrac{\theta}{2}\,e^{-i\varphi} = z$$

Normalized : $|\alpha|^2 + |\beta|^2 = 1$

$$\Rightarrow \alpha = \frac{z}{\sqrt{1+|z|^2}}\,,\quad \beta = \frac{1}{\sqrt{1+|z|^2}}$$ up to common phase factor

$$\Rightarrow \underline{\langle m|z\rangle = \frac{z^{m+1/2}}{\sqrt{1+|z|^2}}}$$

e) $$\langle z|z_0\rangle = \sum_m \langle z|m\rangle\langle m|z_0\rangle = \frac{1+z^* z_0}{\sqrt{(1+|z|^2)(1+|z_0|^2)}}$$

$$\Rightarrow \underline{|\langle z|z_0\rangle|^2 = \frac{1+z^* z_0 + z z_0^* + |z|^2|z_0|^2}{(1+|z|^2)(1+|z_0|^2)}}$$

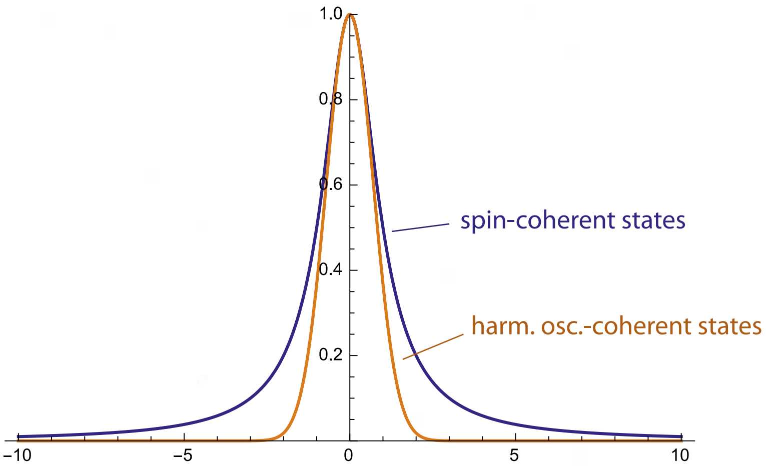
$z_0 = 0$ $$|\langle z|0\rangle|^2 = \frac{1}{1+|z|^2} = \underline{\frac{1}{1+r^2}}$$ $z = r e^{-i\varphi}$

Harmonic oscillator coherent states

$$|\langle z|z_0\rangle|^2 = e^{-|z-z_0|^2}$$

$z_0 = 0$, $z = r e^{-i\varphi}$ $$\Rightarrow |\langle z|0\rangle|^2 = e^{-|z|^2} = \underline{e^{-r^2}}$$

Coherent states, overlap functions $|\langle z|0\rangle|^2$



f) $$I \equiv \int d^2z \,\frac{1}{(1+|z|^2)^2}\, |z\rangle\langle z|$$

$$= \sum_{m,m'} \int d^2z \,\frac{\langle m|z\rangle\langle z|m'\rangle}{(1+|z|^2)^2}\, |m\rangle\langle m'|$$

$$= \sum_{m,m'} \int d^2z \,\frac{z^{m+\frac{1}{2}}\, z^{*\,m'+\frac{1}{2}}}{(1+|z|^2)^3}\, |m\rangle\langle m'|$$

Change to polar coordinates $z = re^{i\varphi}$, $d^2z = r\,d\varphi\,dr$

$$I = \sum_{m,m'} \int_0^\infty dr \,\frac{r^{m+m'+2}}{(1+r^2)^3} \underbrace{\int_0^{2\pi} d\varphi\, e^{i(m-m')\varphi}}_{=\,2\pi\,\delta_{mm'}} |m\rangle\langle m'|$$

$$= 2\pi \sum_m \int_0^\infty \frac{r^{2m+2}}{(1+r^2)^3}\, |m\rangle\langle m|$$

$m = -\frac{1}{2}$: $$\frac{r^{2m+2}}{(1+r^2)^3} = \frac{r}{(1+r^2)^3} = -\frac{1}{4}\frac{d}{dr}\frac{1}{(1+r^2)^2}$$

$$\Rightarrow \int_0^\infty dr \,\frac{r}{(1+r^2)^3} = -\frac{1}{4}\left[\frac{1}{(1+r^2)^2}\right]_0^\infty = \underline{\frac{1}{4}}$$

$m = +\frac{1}{2}$: $$\frac{r^{2m+2}}{(1+r^2)^3} = \frac{r^3}{(1+r^2)^3} = r\left(\frac{1}{(1+r^2)^2} - \frac{1}{(1+r^2)^3}\right)$$

$$= \frac{d}{dr}\left[-\frac{1}{2}\frac{1}{1+r^2} + \frac{1}{4}\frac{1}{(1+r^2)^2}\right]$$

$$\Rightarrow \int_0^\infty dr\,\frac{r^3}{(1+r^2)^3} = \Big[\; -\,''\,- \;\Big]_0^\infty = \underline{\frac{1}{4}}$$

$$I = 2\pi \sum_m \frac{1}{4} |m\rangle\langle m| = \frac{\pi}{2} \mathbb{1}$$

This gives

$$\underline{\int \frac{d^2z}{\pi} \frac{2}{(1+|z|^2)^2} |z\rangle\langle z| = \mathbb{1}}$$

completeness relation for the $j = \frac{1}{2}$ spin coherent states

g) $\hat{H} = \frac{1}{2} \hbar \omega \sigma_z$

$\Rightarrow \hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{2}\omega\sigma_z t}$ time evolution operator

$$\hat{U}(t)|z_0\rangle = \sum_m e^{-\frac{i}{2}\omega\sigma_z t} |m\rangle\langle m|z_0\rangle$$

$$= \sum_m e^{-i\omega m t} |m\rangle\langle m|z_0\rangle \qquad \sigma_z |m\rangle = 2m|m\rangle$$

$$= e^{\frac{i}{2}\omega t} \sum_m \frac{(e^{-i\omega t} z_0)^{m+\frac{1}{2}}}{\sqrt{1+|z_0|^2}} |m\rangle$$

$$= \underline{e^{\frac{i}{2}\omega t} | e^{-i\omega t} z_0\rangle}$$

$= e^{i\alpha(t)} |z(t)\rangle$ with $\underline{\alpha = \frac{1}{2}\omega t}$ and $z(t) = \underline{e^{-i\omega t} z_0}$

Midterm Exam FYS4110/9110, 2015

Solutions

Problem 1

a) Spin compositions

spin ½ x spin ½ = spin 0 + spin 1

with spin 0 and spin 1 defining orthogonal subspaces in the composite Hilbert space

Repeated

spin ½ x (spin ½ x spin ½) = spin ½ x spin 0 + spin ½ x spin 1

$= \underline{\text{spin } 1/2 + \text{spin } 1/2 + \text{spin } 3/2}$

defining three orthogonal subspaces in the full Hilbert space.

b) Scalar products

$$\langle \psi_n | \psi_{n'} \rangle = \frac{1}{3}\left(1 + e^{2\pi i (n'-n)/3} + e^{-2\pi i (n'-n)/3}\right)$$

$$= \frac{1}{3}\left(1 + 2\cos\left(\frac{2\pi}{3}(n'-n)\right)\right)$$

$$n' = n \Rightarrow \cos\left(\frac{2\pi}{3}(n'-n)\right) = \cos 0 = 1$$

$$n' = \pm n \Rightarrow \cos\left(\frac{2\pi}{3}(n'-n)\right) = \cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2}$$

$$\Rightarrow \underline{\langle \psi_n | \psi_{n'} \rangle = \delta_{nn'}} \quad \text{orthogonal for } n \neq n'$$

$$\hat{S}_z |\psi_n\rangle = \frac{\hbar}{2}(1-1-1)|\psi_n\rangle = \underline{-\frac{\hbar}{2}|\psi_n\rangle}$$

Use lowering operator in the spectrum of $\hat{S}_z$

$$\hat{S}_- = \hat{S}_x - i\hat{S}_y = \hat{S}_{-1} + \hat{S}_{-2} + \hat{S}_{-3}$$

For single spin $\hat{S}_- |u\rangle = |d\rangle$, $\hat{S}_- |d\rangle = 0$

For the three spins

$$\hat{S}_-|udd\rangle = \hat{S}_-|dud\rangle = \hat{S}_-|ddu\rangle = |ddd\rangle$$

$$\Rightarrow \hat{S}_-|\psi_n\rangle = \frac{1}{\sqrt{3}}(1+e^{2\pi i n/3}+e^{-2\pi i n/3})|ddd\rangle$$

$$= \frac{1}{\sqrt{3}}(1+2\cos(\tfrac{2\pi n}{3}))|ddd\rangle$$

$\cos(\pm\frac{2\pi}{3}) = -\frac{1}{2} \Rightarrow$

$\underline{\hat{S}_-|\psi_0\rangle = \sqrt{3}\,|ddd\rangle}$ $\quad$ $\underline{\hat{S}_-|\psi_{\pm 1}\rangle = 0}$

This shows that $|\psi_\pm\rangle$ have no component with $s=\frac{3}{2}$

$\Rightarrow$ they are $s=\frac{1}{2}$ states $(\vec{S}^2 = \frac{3}{4}\hbar^2)$

This implies that $|\psi_0\rangle$ is the $s=\frac{3}{2}$ state $(\vec{S}^2 = \frac{15}{4}\hbar^2)$

c) Reduced density operator of spin 1

$$\hat{\rho}_1 = \mathrm{Tr}_{23}\Big(\tfrac{1}{3}(|udd\rangle\langle udd| + |dud\rangle\langle dud| + |ddu\rangle\langle ddu|$$
$$+ e^{2\pi i n/3}(|dud\rangle\langle udd| + |udd\rangle\langle ddu|)$$
$$+ e^{-2\pi i n/3}(|udd\rangle\langle dud| + |ddu\rangle\langle udd|)$$
$$+ e^{4\pi i n/3}|dud\rangle\langle ddu| + e^{-4\pi i n/3}|ddu\rangle\langle dud|)$$

$$= \tfrac{1}{3}|u\rangle\langle u| + \tfrac{2}{3}|d\rangle\langle d|$$

Entanglement entropy for the 1(23) bipartite system

$$S_1 = -\tfrac{1}{3}\log\tfrac{1}{3} - \tfrac{2}{3}\log\tfrac{2}{3} = \log 3 - \tfrac{2}{3}\log 2 = \underline{0.918}$$

max value $S_{1max} = \log 2 = \underline{1}$ $\quad$ (both $\log = \log_2$)

The entanglement entropy is the same for all n, close to but somewhat smaller than the max. value

The symmetry with respect to permuting the spins implies that the other partitions give the same value

d) Measurement of $\hat{S}_{1z}$

The state of spin 1 is projected to $|u\rangle$ or $|d\rangle$ depending on the result

A Result: spin up

$$|\psi_n\rangle \to |udd\rangle = |u\rangle \otimes |d\rangle \otimes |d\rangle$$

product state: no entanglement

B Result: spin down

$$|\psi_n\rangle \to |d\rangle \otimes |\phi_n\rangle$$

$$|\phi_n\rangle = \frac{1}{\sqrt{2}}\left(e^{2\pi i n/3}|ud\rangle + e^{-2\pi i n/3}|du\rangle\right)$$

$$\hat{\rho}_n = |\phi_n\rangle\langle\phi_n| = \frac{1}{2}\left(|ud\rangle\langle ud| + |du\rangle\langle du| + \text{cross terms}\right)$$

Reduced density operators

$$\hat{\rho}_{n1} = \hat{\rho}_{n2} = \frac{1}{2}\left(|u\rangle\langle u| + |d\rangle\langle d|\right) = \frac{1}{2}\mathbb{1}$$

Spin 2 and 3 are now in a maximally mixed state

e) New state

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|uuu\rangle - |ddd\rangle\right)$$

Reduced density operator

$$\hat{\rho}_1 = \mathrm{Tr}_{23}(|\phi\rangle\langle\phi|) = \frac{1}{2}\mathrm{Tr}_{23}\left(|uuu\rangle\langle uuu| + |ddd\rangle\langle ddd| + \text{cross terms}\right)$$

$$= \frac{1}{2}(|u\rangle\langle u| + |d\rangle\langle d|$$

$$= \frac{1}{2}\mathbb{1}$$

Entanglement entropy of partition 1(23)

$S_1 = \underline{\log 2} = 1$ maximal entanglement

The same for the other partitions due to the symmetry of $|\phi\rangle$ under permutation of the spins

f) $|f\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)$, $|b\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$

$|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle)$, $|\ell\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$

$\Rightarrow \quad |u\rangle = \frac{1}{\sqrt{2}}(|f\rangle + |b\rangle) = \frac{1}{\sqrt{2}}(|r\rangle + |\ell\rangle)$

$|d\rangle = \frac{1}{\sqrt{2}}(|f\rangle - |b\rangle) = -\frac{i}{\sqrt{2}}(|r\rangle - |\ell\rangle)$

$\Rightarrow \quad |\phi\rangle = \frac{1}{\sqrt{2}}(|uuu\rangle - |ddd\rangle)$

$= \frac{1}{2}(|bbb\rangle + |f^2b\rangle + |fbf\rangle + |bff\rangle)$

$= \underline{\frac{1}{2}(|rrf\rangle + |\ell\ell f\rangle + |r\ell b\rangle + |\ell r b\rangle)}$

Measurement of S_{2z} or S_{3z} determines S_{1z}

Measurement of S_{2x} and S_{3x}:

outcomes $(bb)_{23} \Rightarrow b_1$

$(fb)_{23} \Rightarrow f_1$

$(bf)_{23} \Rightarrow f_1$

$(ff)_{23} \Rightarrow b_1$

determines <u>uniquely S_{x1}</u>

Measurement of S_{y2} and S_{3x}

outcomes: $(rf)_{23} \Rightarrow r_1$

$(\ell f)_{23} \Rightarrow \ell_1$

$(\ell b)_{23} \Rightarrow r_1$

$(rb)_{23} \Rightarrow \ell_1$

determines <u>uniquely S_{y1}</u>

Problem 2

a) Total spin $\vec{S} = \frac{\hbar}{2}(\vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{\sigma}) \equiv \frac{\hbar}{2}(\vec{\Sigma}_A + \vec{\Sigma}_B)$

$$\vec{S}^2 = \frac{\hbar^2}{2}(3\,\mathbb{1}\otimes\mathbb{1} + \vec{\Sigma}_A \cdot \vec{\Sigma}_B)$$

$$= \frac{\hbar^2}{2}\left(3\mathbb{1} + \sum_{k=1}^{3} \sigma_k \otimes \sigma_k\right)$$

$$\sigma_k \otimes \sigma_k |\psi_a\rangle = -|\psi_a\rangle$$

$$\sigma_z \otimes \sigma_z |\psi_s\rangle = -|\psi_s\rangle$$

$$\sigma_x \otimes \sigma_x |\psi_s\rangle = \sigma_y \otimes \sigma_y |\psi_s\rangle = |\psi_s\rangle$$

The three cases

I $\quad \langle \vec{S}^2 \rangle_1 = \frac{\hbar^2}{2}(3-3) = \underline{0}$

II $\quad \langle \vec{S}^2 \rangle_2 = \frac{\hbar^2}{2}(3+1) = \underline{2\hbar^2}$

III $\quad \langle \vec{S}^2 \rangle_3 = \frac{1}{2}(\langle \vec{S}^2 \rangle_1 + \langle \vec{S}^2 \rangle_2) = \underline{\hbar^2}$

$\hat{\rho}_1$ is a spin 0 state, $\hat{\rho}_2$ is a spin 1 state and $\hat{\rho}_3$ is a mixed state composed of spin 0 and 1

$\Rightarrow$ Only $\hat{\rho}_1$ is rotationally invariant

b) Reduced density operators

cross terms

$$\hat{\rho}_1^A = \mathrm{Tr}_B\left[\tfrac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+| - |+-\rangle\langle-+| - |-+\rangle\langle+-|)\right]$$

$$= \tfrac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|)_A = \underline{\tfrac{1}{2}\mathbb{1}_A}$$

Since the cross terms do not contribute:

$$\left.\begin{array}{l} \hat{\rho}_2^A = \hat{\rho}_3 = \hat{\rho}_1 = \underline{\tfrac{1}{2}\mathbb{1}_A} \\ \text{Similarly } \hat{\rho}_1^B = \hat{\rho}_2^B = \hat{\rho}_3^B = \underline{\tfrac{1}{2}\mathbb{1}_B} \end{array}\right\} \text{maximally mixed}$$

$\hat{\rho}_1$ and $\hat{\rho}_2$ are pure states $\Rightarrow$ entropies $\underline{S_1 = S_2 = 0}$

$\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2)$ is mixed with probabilities $p_1 = p_2 = \frac{1}{2}$

$\Rightarrow$ entropy $S_3 = -p_1 \log p_1 - p_2 \log p_2 = \underline{\log 2}$

Entropies of subsystems

$S_1^A = S_2^A = S_3^A = \underline{\log 2}$, same for B

Inequality: $S_{max} \geq \max\{S_A, S_B\}$

I and II: not satisfied

III: satisfied as equality

Degree of entanglement

I and II are pure states,

entanglement entropies $\underline{S_1^A = S_2^A = \log 2}$, same for B

maximally entangled

$\underline{\text{III}}$: $\hat{\rho}_3 = \frac{1}{2}(\hat{\rho}_1 + \hat{\rho}_2) = \frac{1}{2}(|+-\rangle\langle+-| + |-+\rangle\langle-+|)$

$= \frac{1}{2}(|+\rangle\langle+| \otimes |-\rangle\langle-| + |-\rangle\langle-| \otimes |+\rangle\langle+|)$

mixture of product states $\Rightarrow$ separable

no entanglement

c) $|\theta\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \Rightarrow$

$\hat{S}_\theta|\theta\rangle = (\cos\theta\, S_z + \sin\theta\, S_x)(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle)$

$= \frac{\hbar}{2}\left[(\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2})|+\rangle + (\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2})|-\rangle\right]$

$= \underline{\frac{\hbar}{2}(\cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle) = |\theta\rangle}$

$$P_A = Tr_A(\hat{\rho}_A \hat{P}(\theta)) = \langle\theta|\tfrac{1}{2}\mathbb{1}_A|\theta\rangle = \underline{\tfrac{1}{2}}$$

This is valid for all three cases I, II and III, it means that the probabilities for spin up and down are equal for any direction θ.

d) Joint probabilities

$$P(\theta,\theta') = Tr(\hat{\rho}\,\hat{P}(\theta)\otimes\hat{P}(\theta'))$$

$$= \langle\theta,\theta'|\hat{\rho}|\theta,\theta'\rangle \qquad |\theta;\theta'\rangle = |\theta\rangle\otimes|\theta'\rangle$$

$$\langle +-|\theta,\theta'\rangle = \langle +|\theta\rangle\langle -|\theta'\rangle = \cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2}$$

$$\langle -+|\theta,\theta'\rangle = \langle -|\theta\rangle\langle +|\theta'\rangle = \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}$$

Case I:

$$P_1(\theta,\theta') = \tfrac{1}{2}\big[\langle\theta\theta'|+-\rangle\langle +-|\theta\theta'\rangle + \langle\theta\theta'|-+\rangle\langle -+|\theta\theta'\rangle$$

$$- \langle\theta\theta'|+-\rangle\langle -+|\theta\theta'\rangle - \langle\theta\theta'|-+\rangle\langle +-|\theta\theta'\rangle\big]$$

$$= \tfrac{1}{2}\big[\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta'}{2} + \sin^2\tfrac{\theta}{2}\cos^2\tfrac{\theta'}{2} - 2\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}\sin\tfrac{\theta'}{2}\big]$$

$$= \tfrac{1}{2}\left(\cos\tfrac{\theta}{2}\sin\tfrac{\theta'}{2} - \sin\tfrac{\theta}{2}\cos\tfrac{\theta'}{2}\right)^2$$

$$= \underline{\tfrac{1}{2}\sin^2\tfrac{\theta-\theta'}{2}}$$

Similar evaluations for case II and III

$$\underline{P_2(\theta,\theta') = \tfrac{1}{2}\sin^2\tfrac{\theta+\theta'}{2}}, \quad \underline{P_3(\theta,\theta') = \tfrac{1}{4}\left(\sin^2\tfrac{\theta-\theta'}{2} + \sin^2\tfrac{\theta+\theta'}{2}\right)}$$

f) Experimental quantities

$$P^A_{exp}(\theta) = \underline{\frac{n_{++}+n_{+-}}{N}}, \quad P^B_{exp}(\theta) = \underline{\frac{n_{++}+n_{-+}}{N}}$$

$$P_{exp}(\theta,\theta') = \underline{\frac{n_{++}}{N}}$$

e) Plots of the function $F(\theta, \theta')$

Left: Plot of the curves $F(\theta, \theta/2)$ for cases I, II, III

Right: 3D plots of $F(\theta, \theta')$

Cases I and II: Bell's inequality broken (negative F, colored red in 3D plot

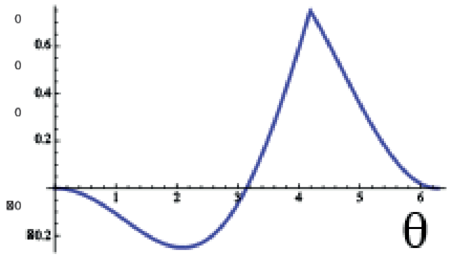
Case III: Bell's inequality unbroken (F positive)

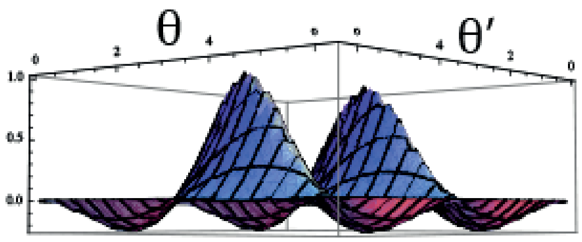
Results consistent with b): I an II entangled state, III non-entangled

$\theta' = 0.5\,\theta$

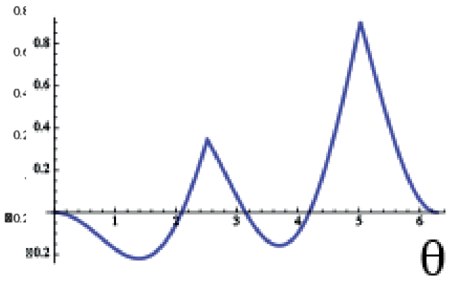
3D plot

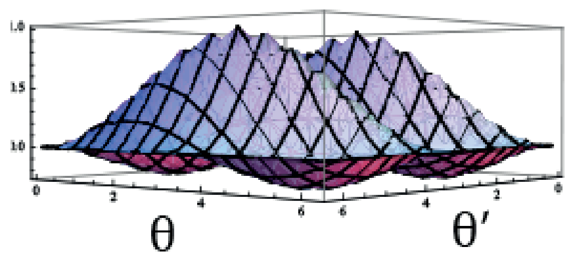
Case I



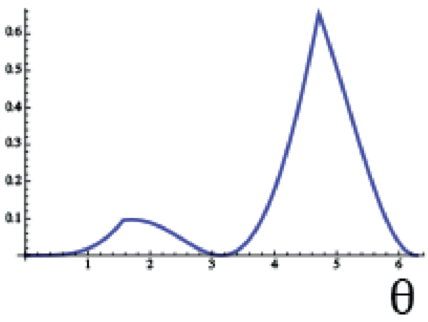


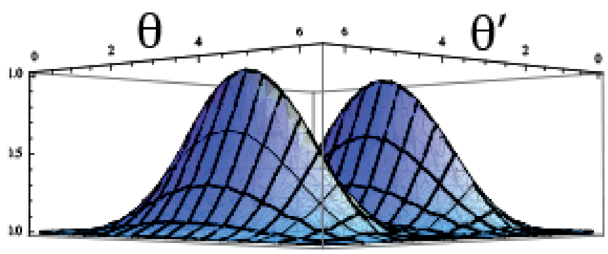
Case II





Case III





Midterm Exam FYS4110, 2016
Solutions

Problem 1

a) The expectation values P_1, P_2 and P_{12} determine the probabilities of detecting photons with the given polarization, respectively at detector 1, detector 2 and at both detectors. This implies the following correspondences, $\frac{n_1}{N} \approx P_1$ for large N, similarly $\frac{n_2}{N} \approx P_2$ and $\frac{n_{12}}{N} \approx P_{12}$.

b) Density operator, two-photon system

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}(|HV\rangle\langle HV| + |VH\rangle\langle VH| + e^{i\chi}|VH\rangle\langle HV| + e^{-i\chi}|HV\rangle\langle VH|) \tag{1}$$

Reduced density operators

$$\begin{aligned}
\hat{\rho}_1 &= Tr_2\hat{\rho} = \langle H_2|\hat{\rho}|H_2\rangle + \langle V_2|\hat{\rho}|V_2\rangle = \frac{1}{2}(|H\rangle\langle H| + |V\rangle\langle V|)_1 = \frac{1}{2}\mathbb{1}_1 \\
\hat{\rho}_2 &= Tr_1\hat{\rho} = \langle H_1|\hat{\rho}|H_1\rangle + \langle V_1|\hat{\rho}|V_1\rangle = \frac{1}{2}(|H\rangle\langle H| + |V\rangle\langle V|)_2 = \frac{1}{2}\mathbb{1}_2
\end{aligned} \tag{2}$$

Both reduced density operators have maximum von Neuman entropy $S_{1/2} = -Tr\hat{\rho}_{1/2}\log\hat{\rho}_{1/2} = \log 2$. Since the two-photon system is in a pure state, $S_{1/2}$ is equal to the entanglement entropy, which gives the measure of the degree of entanglement between the two photons. Thus, the photon pairs have maximum entanglement for all values of the phase angle χ.

c) Since the reduced density operators are independent of χ, the results for P_1 and P_2 are the same in the three cases,

$$\begin{aligned}
P_1(\theta_1) &= \mathrm{Tr}(\hat{\rho}\hat{P}_1(\theta_1)) = \mathrm{Tr}_1(\hat{\rho}_1\hat{P}_1(\theta_1)) = \frac{1}{2}\,\mathrm{Tr}\,\hat{P}_1(\theta_1) = \frac{1}{2}\langle\theta_1|\theta_1\rangle = \frac{1}{2} \\
P_2(\theta_2) &= \mathrm{Tr}(\hat{\rho}\hat{P}_2(\theta_2)) = \mathrm{Tr}_2(\hat{\rho}_2\hat{P}_2(\theta_2)) = \frac{1}{2}\,\mathrm{Tr}\,\hat{P}_2(\theta_2) = \frac{1}{2}\langle\theta_2|\theta_2\rangle = \frac{1}{2}
\end{aligned} \tag{3}$$

The probabilities P_1 and P_2 are independent of the polarization angles.

The joint probability is given by

$$P_{12}(\theta_1,\theta_2) = \mathrm{Tr}(\hat{\rho}|\theta_1\theta_2\rangle\langle\theta_1\theta_2|) = |\langle\psi|\theta_1\theta_2\rangle|^2\,, \quad |\theta_1\theta_2\rangle = |\theta_1\rangle\otimes|\theta_2\rangle \tag{4}$$

case I: $\chi = \pi$

$$\begin{aligned}
|\psi_I\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) \\
\Rightarrow \quad \langle\psi_I|\theta_1\theta_2\rangle &= \frac{1}{\sqrt{2}}(\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2)) \\
&= -\frac{1}{\sqrt{2}}\sin(\theta_1-\theta_2) \\
\Rightarrow \quad P_{12}(\theta_1,\theta_2) &= \frac{1}{2}\sin^2(\theta_1-\theta_2)
\end{aligned} \tag{5}$$

case II: $\chi = 0$

$$\begin{aligned} |\psi_{II}\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) \\ \Rightarrow \quad \langle\psi_{II}|\theta_1\theta_2\rangle &= \frac{1}{\sqrt{2}}(\cos(\theta_1)\sin(\theta_2) + \sin(\theta_1)\cos(\theta_2)) \\ &= -\frac{1}{\sqrt{2}}\sin(\theta_1+\theta_2) \\ \Rightarrow \quad P_{12}(\theta_1,\theta_2) &= \frac{1}{2}\sin^2(\theta_1+\theta_2) \qquad (6) \end{aligned}$$

case III: $\chi = \pi/2$

$$\begin{aligned} |\psi_{III}\rangle &= \frac{1}{\sqrt{2}}(|HV\rangle + i|VH\rangle) \\ \Rightarrow \quad \langle\psi_{III}|\theta_1\theta_2\rangle &= \frac{1}{\sqrt{2}}(\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2)) \\ \Rightarrow \quad P_{12}(\theta_1,\theta_2) &= \frac{1}{2}(\cos^2(\theta_1)\sin^2(\theta_2) + \sin^2(\theta_1)\cos^2(\theta_2)) \\ &= \frac{1}{4}(\sin^2(\theta_1-\theta_2) + \sin^2(\theta_1+\theta_2)) \qquad (7) \end{aligned}$$

d) The result (7) is the same as half the sum of the corresponding results for the cases I and II. This means that the expression for P_{12} in case III is the same as for the density operator

$$\hat{\rho}'_{III} = \frac{1}{2}(\hat{\rho}_I + \hat{\rho}_{II}) = \frac{1}{2}(|\psi_I\rangle\langle\psi_I| + |\psi_{II}\rangle\langle\psi_{II}|) = \frac{1}{2}(|H\rangle\langle H| \otimes |V\rangle\langle V| + |V\rangle\langle V| \otimes |H\rangle\langle H|) \qquad (8)$$

which is a separable (unentangled) state.

e) Define the function

$$F(\theta) = F(0,\theta,2\theta) = P_{12}(\theta,2\theta) - |P_{12}(0,\theta) - P_{12}(0,2\theta)| \qquad (9)$$

This function should be non-negative if Bell's inequality is satisfied. Three plots are shown of this function, corresponding to the three cases I, I, III. In case I and II the curves do not satisfy the inequality, in accordance with the expectation that when the two-photon state is entangled Bell's inequality is nor respected. In case III the function is non-negative, which means that the Bell inequality is unbroken. This can be understood as due to the fact that the same expression for $F(\theta)$ can be found for a separable (unentangled) two-photon state. Since also in case III the state is maximally entangled, the Bell inequality studied here can not be sufficient general to register entanglement for all values of χ.

f) Results with detector 2 projecting on the new polarization states with $\phi = \pm\pi/4$.

The two-photon polarization state corresponds to case III ($\chi = \pi/2$).

Polarization state of the two projectors,

$$|\theta_1\theta_{\phi 2}\rangle = \cos\theta_1\sin\theta_2 e^{-i\phi}|HV\rangle + \sin\theta_1\cos\theta_2 e^{i\phi}|VH\rangle + (terms\ |HH\rangle, |VV\rangle) \qquad (10)$$

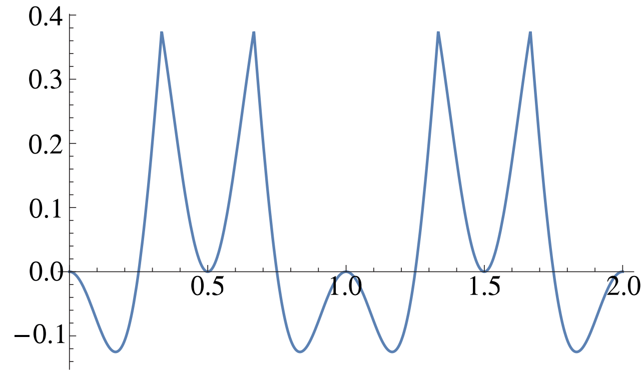
The joint probability is now

$$\begin{aligned} P_{12}(\theta_1,\theta_2) &= |\langle\psi_{III}|\theta_1\theta_{\phi 2}\rangle|^2 \\ &= \left|\frac{1}{\sqrt{2}}(e^{-i\phi}\cos\theta_1\sin\theta_2 - ie^{i\phi}\sin\theta_1\cos\theta)\right|^2 \\ &= \frac{1}{4}\big((1+\sin 2\phi)\sin^2(\theta_1+\theta_2) + (1-\sin 2\phi)\sin^2(\theta_1-\theta_2)\big) \qquad (11) \end{aligned}$$

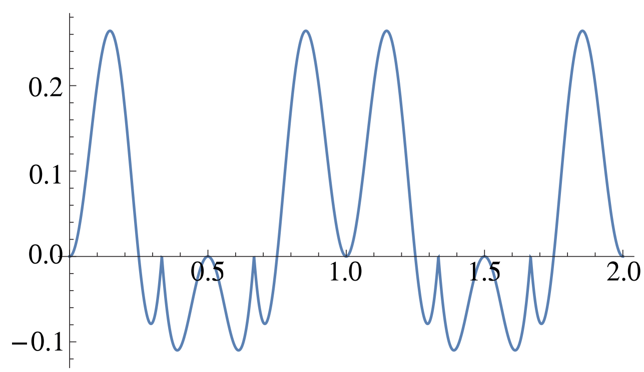
Problem1e)

Bell's inequality: Plots of F(0,θ,2θ)

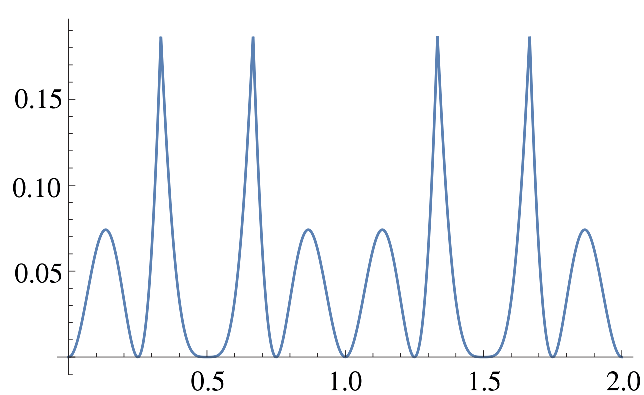
Case I: $\chi=\pi$



Case II: $\chi=0$



Case III: $\chi=\pi/2$



Case A:

$$\phi = \pi/4 \Rightarrow \sin 2\phi = 1 \Rightarrow P_{12} = \frac{1}{2}\sin^2(\theta_1 + \theta_2) \tag{12}$$

Case B:

$$\phi = -\pi/4 \Rightarrow \sin 2\phi = -1 \Rightarrow P_{12} = \frac{1}{2}\sin^2(\theta_1 - \theta_2) \tag{13}$$

We note that $P_{12}(\theta_1, \theta_2)$ in case A is the the same function of θ_1 and θ_2 as earlier found in case I (Eq. (5)). Similarly $P_{12}(\theta_1, \theta_2)$ in case B is the the same function as earlier found in case II (Eq. (6)). In both cases Bell's inequality is broken, and similarly this will be true in cases A and B. Consequently breaking of Bell's inequality is found also for the state $|\psi_{III}\rangle$, but only if one of the detectors register non-linear photon polarization.

2 Atom-photon interactions in a microcavity

a) Action of $\hat{H}$ on the basis states

$$\begin{aligned}
\hat{H}|g,1\rangle &= (\frac{1}{2}\hbar\omega - i\gamma\hbar)|g,1\rangle + \frac{1}{2}\lambda|e,0\rangle \\
\hat{H}|e,0\rangle &= \frac{1}{2}\hbar\omega|e,0\rangle + \frac{1}{2}\lambda|g,1\rangle \\
\hat{H}|g,0\rangle &= -\frac{1}{2}\hbar\omega|g,0\rangle
\end{aligned} \tag{14}$$

The ground state $|g,0\rangle$ is disconnected from the other states and can be disregarded. Extracting the matrix elements of $\hat{H}$ from (14) we find that the Hamiltonian, restricted to the subspace spanned by the vectors $|g,1\rangle$ and $|e,0\rangle$, takes the matrix form

$$H = \frac{1}{2}\hbar(\omega - i\gamma)\mathbb{1} + \frac{1}{2}\hbar\begin{pmatrix} i\gamma & \lambda \\ \lambda & -i\gamma \end{pmatrix} \tag{15}$$

b) The time evolution operator is

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{\hbar}\hat{H}t} = e^{-\frac{i}{2}(\omega - i\gamma)t}e^{-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t} \tag{16}$$

with $\boldsymbol{\Omega} = \frac{1}{2}(\lambda\mathbf{i} + i\gamma\mathbf{k})$. The second term can be expanded in powers of the Pauli matrix $\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}/\Omega$,

$$\begin{aligned}
e^{-i\boldsymbol{\Omega}\cdot\boldsymbol{\sigma}t} &= (1 - \frac{1}{2}\Omega^2 t^2 + \frac{1}{4!}\Omega^4 t^4 ...)\mathbb{1} \\
&\quad -i\frac{\boldsymbol{\omega}}{\Omega}\cdot\boldsymbol{\sigma}(\Omega t - \frac{1}{3!}\Omega^3 t^3 + ...) \\
&= \cos(\Omega t)\mathbb{1} - i\frac{\boldsymbol{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}\sin(\Omega t)
\end{aligned} \tag{17}$$

where we have exploited the property of Pauli matrices that even powers are proportional to the identity and odd order are proportional to the Pauli matrix. From this follows the result

$$\hat{\mathcal{U}}(t) = e^{-\frac{i}{2}(\omega - i\gamma)t}(\cos(\Omega t)\mathbb{1} - i\sin(\Omega t)\,\frac{\boldsymbol{\Omega}}{\Omega}\cdot\boldsymbol{\sigma}) \tag{18}$$

$\boldsymbol{\Omega} = \frac{1}{2}(\lambda\mathbf{i} + i\gamma\mathbf{k})$ gives $\Omega^2 = \frac{1}{4}(\lambda^2 - \gamma^2)$ and $\Omega = \frac{1}{2}\sqrt{\lambda^2 - \gamma^2}$, which is real and positive when $\lambda > \gamma$.

c) In matrix form the time dependent wave function is

$$\begin{aligned}\psi(t) &= \hat{\mathcal{U}}(t)\psi(0) \\ &= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix}\cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t & -i\frac{\lambda}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t & \cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\ &= e^{-\frac{1}{2}(i\omega+\gamma)t}\begin{pmatrix}\cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t \\ -i\frac{\lambda}{2\Omega}\sin\Omega t\end{pmatrix} \qquad (19)\end{aligned}$$

In bra-ket form this gives

$$|\psi(t)\rangle = e^{-\frac{1}{2}(i\omega+\gamma)t}\left((\cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t)|e,0\rangle - i\frac{\lambda}{2\Omega}\sin\Omega t|g,1\rangle\right) \qquad (20)$$

d) Assuming $\mathrm{Tr}\,\hat{\rho}_{cav}=1$ we find

$$\begin{aligned}f(t) &= 1-\mathrm{Tr}\,\hat{\rho}(t) \\ &= 1-\langle\psi(t)|\psi(t)\rangle \\ &= 1-e^{-\gamma t}\left(\frac{\lambda^2}{4\Omega^2}-\frac{\gamma^2}{4\Omega^2}\cos(2\Omega t)+\frac{\gamma}{2\Omega}\sin(2\Omega t)\right) \qquad (21)\end{aligned}$$

When the photon escapes through the walls, the system inside the cavity ends up in the state $|g,0\rangle$. The term added to the density matrix $\hat{\rho}$ takes care of this in such a way that the sum of the probabilities for the atom to be in one of the states $|e\rangle$ and $|g\rangle$ is constant, equal to 1.

e) Occupation probabilities for the atom; the excited state

$$\begin{aligned}p_e(t) &= \langle e,0|\hat{\rho}_{tot}(t)|e,0\rangle \\ &= \langle e,0|\hat{\rho}(t)|e,0\rangle \\ &= |\langle\psi(t)|e,0\rangle|^2 \\ &= e^{-\gamma t}(\cos\Omega t+\frac{\gamma}{2\Omega}\sin\Omega t)^2 \\ &= e^{-\gamma t}(\frac{\lambda^2}{8\Omega^2}+\frac{\lambda^2-2\gamma^2}{8\Omega^2}\cos(2\Omega t)+\frac{\gamma}{2\Omega}\sin(2\Omega t) \qquad (22)\end{aligned}$$

and the ground state

$$p_g(t) = 1-p_e(t) \qquad (23)$$

The probability for one photon being present in the cavity is

$$\begin{aligned}p_{ph}(t) &= \langle g,1|\hat{\rho}(t)|g,1\rangle \\ &= |\langle\psi(t)|g,1\rangle|^2 \\ &= \frac{\lambda^2}{8\Omega^2}e^{-\gamma t}(1-\cos(2\Omega t)) \qquad (24)\end{aligned}$$

f) Eigenvalues of $\hat{\rho}_{cav}(t)$,

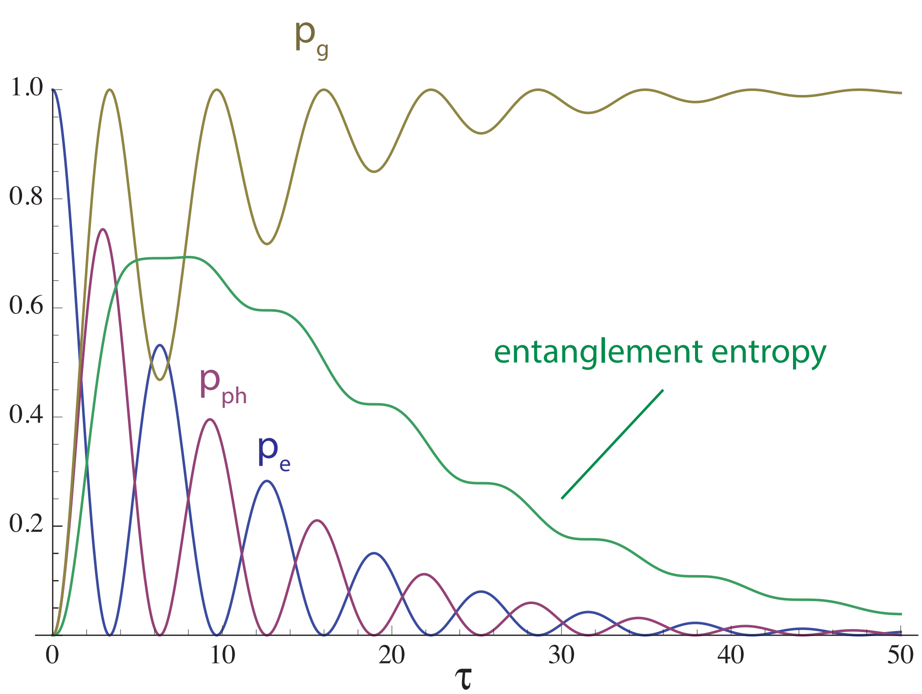
$$\begin{aligned}\hat{\rho}_{cav}(t) &= |\psi(t)\rangle\langle\psi(t)|+f(t)|g,0\rangle\langle g,0| \\ &= \langle\psi(t)|\psi(t)\rangle|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)|+f(t)|g,0\rangle\langle g,0| \\ &= (1-f(t))|\tilde{\psi}(t)\rangle\langle\tilde{\psi}(t)|+f(t)|g,0\rangle\langle g,0| \qquad (25)\end{aligned}$$

where $|\tilde{\psi}(t)\rangle$ is normalized to 1. Since this state is orthogonal to the normalized state $|g, 0\rangle$, the above expression gives the spectral decomposition of $\hat{\rho}_{cav}$, with eigenvalues $f(t)$ and $1 - f(t)$. The corresponding von Neuman entropy is

$$S = -f \log f - (1 - f) \log(1 - f) \tag{26}$$

with f given by (21). With the cavity system viewed as a part of a larger system in a pure state, which includes also the photon states of the escaped photon, the above expression for S can be identified as the entanglement entropy of the larger, composite system.

Problem 2 e) og f)
Occupation probabilities and entanglement entropy



Fys 4110 Midterm exam 2017 Solutions

Problem 1.

a) Hamiltonian: $H = -\frac{\hbar\omega_0}{2}\sigma_z - \frac{\hbar\omega_1}{2}(\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y)$

We transform to a rotating frame with angular velocity ω (same as driving field).

Time dependent unitary transform $T(t) = e^{-\frac{i\omega}{2}t\sigma_z}$

Transformed state $|\psi'\rangle = T(t)|\psi\rangle$

Hamiltonian $H' = THT^\dagger + i\hbar\frac{dT}{dt}T^\dagger$

Using the relations

$$e^{-\frac{i\omega}{2}t\sigma_z}\,\sigma_x\,e^{i\frac{\omega}{2}t\sigma_z} = \cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y$$

$$e^{-i\frac{\omega}{2}t\sigma_z}\,\sigma_y\,e^{i\frac{\omega}{2}t\sigma_z} = \cos\omega t\,\sigma_y - \sin\omega t\,\sigma_x$$

we get a time-independent Hamiltonian

$$H' = \frac{\hbar}{2}(\omega-\omega_0)\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Define: $\Omega = \sqrt{(\omega-\omega_0)^2+\omega_1^2}$

$$\cos\theta = \frac{\omega_0-\omega}{\Omega} \qquad \sin\theta = \frac{\omega_1}{\Omega}$$

$$H' = -\frac{1}{2}\hbar\Omega(\cos\theta\,\sigma_z + \sin\theta\,\sigma_x)$$

This gives the time evolution

$$U'(t) = e^{-\frac{i}{\hbar}H't} = \cos\frac{\Omega t}{2}\mathbb{1} + i\sin\frac{\Omega t}{2}(\cos\theta\,\sigma_z + \sin\theta\,\sigma_x)$$

Transform back: $U(t) = T(t)^\dagger U'(t) \underbrace{T(0)}_{\mathbb{1}}$ ②

If $|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$ with $\overbrace{c_0(0)=1 \text{ and } c_1(0)=0}^{\text{Ground state}}$

we get
$$c_0(t) = \left(\cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta\right)e^{-\frac{i\omega t}{2}}$$
$$c_1(t) = i\sin\frac{\Omega t}{2}\sin\theta\, e^{\frac{i\omega t}{2}}$$

The probability to find the excited state is
$$P_1(t) = |c_1(t)|^2 = \sin^2\frac{\Omega t}{2}\sin^2\theta$$

b) Hamiltonian:
$$H = \underbrace{\tfrac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega a^\dagger a}_{H_0} + \underbrace{i\hbar\lambda(a^\dagger\sigma_- - a\sigma_+)}_{H_1}$$

The eigenstates of H_0:
$$H_0|\pm,n\rangle = \underbrace{\hbar(n\omega \pm \tfrac{1}{2}\omega_0)}_{E_{\pm,n}}|\pm,n\rangle$$

The ground state is unaffected by interaction: $H_1|-,0\rangle = 0$

For the excited states we have:
$$H_1|+,n\rangle = i\hbar\lambda\sqrt{n+1}\,|-,n+1\rangle$$
$$H_1|-,n+1\rangle = -i\hbar\lambda\sqrt{n+1}\,|+,n\rangle$$

$\Rightarrow$ H_1 mixes only pairs of states and the full H consists of 2×2 blocks on the diagonal.

In the space $\{|+,n\rangle, |-,n+1\rangle\}$ we have
$$H_n = \frac{1}{2}\hbar\begin{pmatrix}\Delta & -ig_n \\ ig_n & -\Delta\end{pmatrix} + \epsilon_n\mathbb{1}$$

$\Delta = \omega_0 - \omega \qquad g_n = 2\lambda\sqrt{n+1} \qquad \epsilon_n = (n+\tfrac{1}{2})\hbar\omega$

Defining $\Omega_n = \sqrt{\Delta^2 + g_n^2}$ $\cos\theta_n = \frac{\Delta}{\Omega_n}$ $\sin\theta_n = \frac{g_n}{\Omega_n}$

$$H_n = \tfrac{1}{2}\hbar\Omega_n(\cos\theta_n\,\sigma_z + \sin\theta_n\,\sigma_y) + \epsilon_n \mathbb{1}$$

The eigenstates are $|\psi_n^+\rangle = \cos\frac{\theta_n}{2}|+,n\rangle + i\sin\frac{\theta_n}{2}|-,n+1\rangle$

$|\psi_n^-\rangle = i\sin\frac{\theta_n}{2}|+,n\rangle + \cos\frac{\theta_n}{2}|-,n+1\rangle$

with eigenvalues $E_n^\pm = \epsilon_n \pm \tfrac{1}{2}\hbar\Omega_n$

Using this we can now find the time evolution of a general state in the $\{|+,n\rangle, |-,n+1\rangle\}$ space:

$$|\psi(0)\rangle = c_n^+(0)|+,n\rangle + c_n^-(0)|-,n+1\rangle$$
$$= d_n^+|\psi_n^+\rangle + d_n^-|\psi_n^-\rangle$$
$$\xrightarrow{\text{time}} d_n^+ e^{-\frac{i}{\hbar}E_n^+ t}|\psi_n^+\rangle + d_n^- e^{-\frac{i}{\hbar}E_n^- t}|\psi_n^-\rangle$$
$$= c_n^+(t)|+,n\rangle + c_n^-(t)|-,n+1\rangle$$

With the initial state $|-,n+1\rangle$ we have $c_n^+(0)=0$, $c_n^-(0)=1$

and get
$$c_n^+(t) = -e^{-\frac{i}{\hbar}\epsilon_n t}\sin\theta_n \sin\frac{\Omega_n t}{2}$$
$$c_n^-(t) = -e^{-\frac{i}{\hbar}\epsilon_n t}\left(\cos\frac{\Omega_n t}{2} + i\cos\theta_n \sin\frac{\Omega_n t}{2}\right)$$

Probability for the excited state is

$$P_+(t) = |c_n^+(t)|^2 = \sin^2\theta_n \sin^2\frac{\Omega_n t}{2}$$

Comparing to the Rabi problem, this is the same provided we identify $\omega_1 \leftrightarrow g_n$

g) We have $|\Psi(t)\rangle = c_n^+(t)|+,n\rangle + c_n^-(t)|-,n+1\rangle$ with $c_n^\pm(t)$ given in b).

Density matrix: $\rho = |\Psi(t)\rangle\langle\Psi(t)|$

$$= |c_n^+(t)|^2 |+,n\rangle\langle +,n| + c_n^+(t)\, c_n^-(t)^* |+,n\rangle\langle -,n+1|$$

$$+ \; c_n^+(t)^* c_n^-(t) |-,n+1\rangle\langle +,n| + |c_n^-(t)|^2 |-,n+1\rangle\langle -,n+1|$$

Tracing over the photon mode:

$$\rho_{TLS} = \mathrm{Tr}_{photon}\, \rho = \sum_m \langle m|\rho|m\rangle = |c_n^+(t)|^2 |+\rangle\langle +| + |c_n^-(t)|^2 |-\rangle\langle -|$$

We have $|c_n^+(t)|^2 = \sin^2\theta_n \sin^2\frac{\Omega_n t}{2} = p^+$

$|c_n^-(t)|^2 = 1 - \sin^2\theta_n \sin^2\frac{\Omega_n t}{2} = p^-$

Entanglement entropy:

$$S = -\mathrm{Tr}\, \rho_{TLS} \ln \rho_{TLS} = -p^+ \ln p^+ - p^- \ln p^-$$

$$= -\sin^2\theta_n \sin^2\frac{\Omega_n t}{2} \ln\left(\sin^2\theta_n \sin^2\frac{\Omega_n t}{2}\right)$$

$$-\left(1 - \sin^2\theta_n \sin^2\frac{\Omega_n t}{2}\right) \ln\left(1 - \sin^2\theta_n \sin^2\frac{\Omega_n t}{2}\right)$$

Maximal entropy when p^+ and p^- are as equal as possible.

If $\sin^2\theta_n > \frac{1}{2}$, $\theta_n > \pi/4$ we can get $p^+ = p^- = \frac{1}{2}$

with $S_{max} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \ln 2$

This happens when $\sin^2\theta_n \sin^2\frac{\Omega_n t}{2} = \frac{1}{2}$

$$\Rightarrow t = \frac{2}{\Omega_n}\arcsin\left[\frac{1}{\sqrt{2}\sin\theta_n}\right] = \frac{2}{\Omega_n}\arcsin\left[\frac{\Omega_n}{\sqrt{2}\,g_n}\right]$$

If $\sin^2\theta_n < \frac{1}{2}$ we have $p^+ < \frac{1}{2}$ and maximal when $\frac{\Omega_n t}{2} = \frac{\pi}{2} + m\pi \quad (m \in \mathbb{Z})$

$$p^+_{max} = \sin^2\theta_n \quad , \quad S_{max} = -\sin^2\theta_n \ln\sin^2\theta_n - \cos^2\theta_n \ln\cos^2\theta_n$$

d) For the Rabi model (in rotating frame):

$$|\psi(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle$$

$$c_0(t) = \cos\frac{\Omega t}{2} + i\sin\frac{\Omega t}{2}\cos\theta \qquad c_1(t) = i\sin\frac{\Omega t}{2}\sin\theta$$

This is a pure state and the Bloch vector has components

$$m_x^R = 2\,\mathrm{Re}(c_0^* c_1) = \sin 2\theta \sin^2\frac{\Omega t}{2}$$

$$m_y^R = 2\,\mathrm{Im}(c_0^* c_1) = \sin\theta \sin\Omega t$$

$$m_z^R = |c_0|^2 - |c_1|^2 = \cos^2\frac{\Omega t}{2} + \sin^2\frac{\Omega t}{2}\cos^2\theta - \sin^2\frac{\Omega t}{2}\sin^2\theta$$

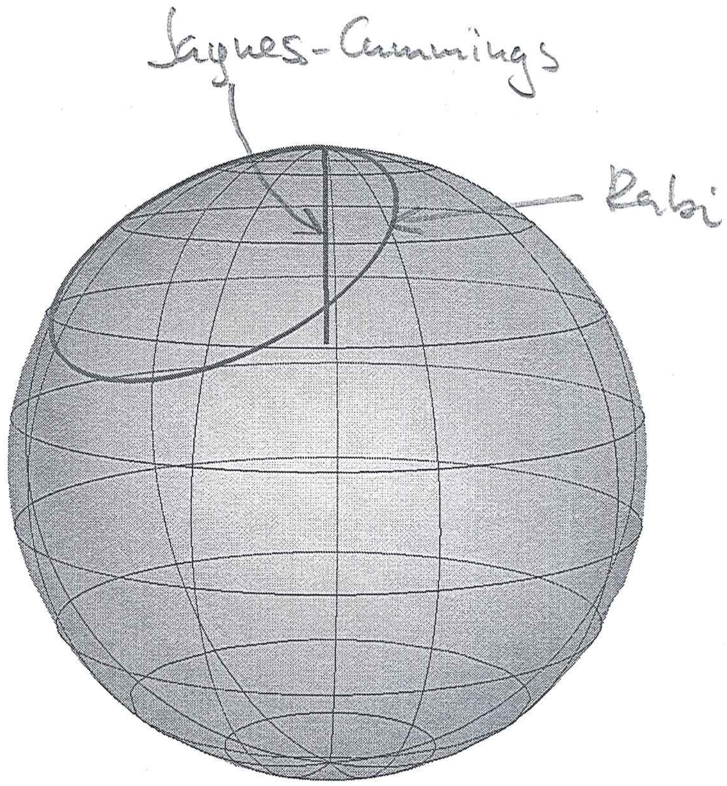
$$= 1 - 2\sin^2\theta \sin^2\frac{\Omega t}{2}$$

For the JC model we use $\rho_{TLS} = \frac{1}{2}(\mathbb{1} + \vec{m}^{JC}\cdot\vec{\sigma})$

$$\rho_{TLS} = p^-|-\rangle\langle -| + p^+|+\rangle\langle +| = \frac{1}{2}(\mathbb{1} + (p^- - p^+)\sigma_z)$$

$$\Rightarrow \quad m_x^{JC} = m_y^{JC} = 0$$

$$m_z^{JC} = p^- - p^+ = 1 - 2\sin^2\theta_n \sin^2\frac{\Omega t}{2}$$

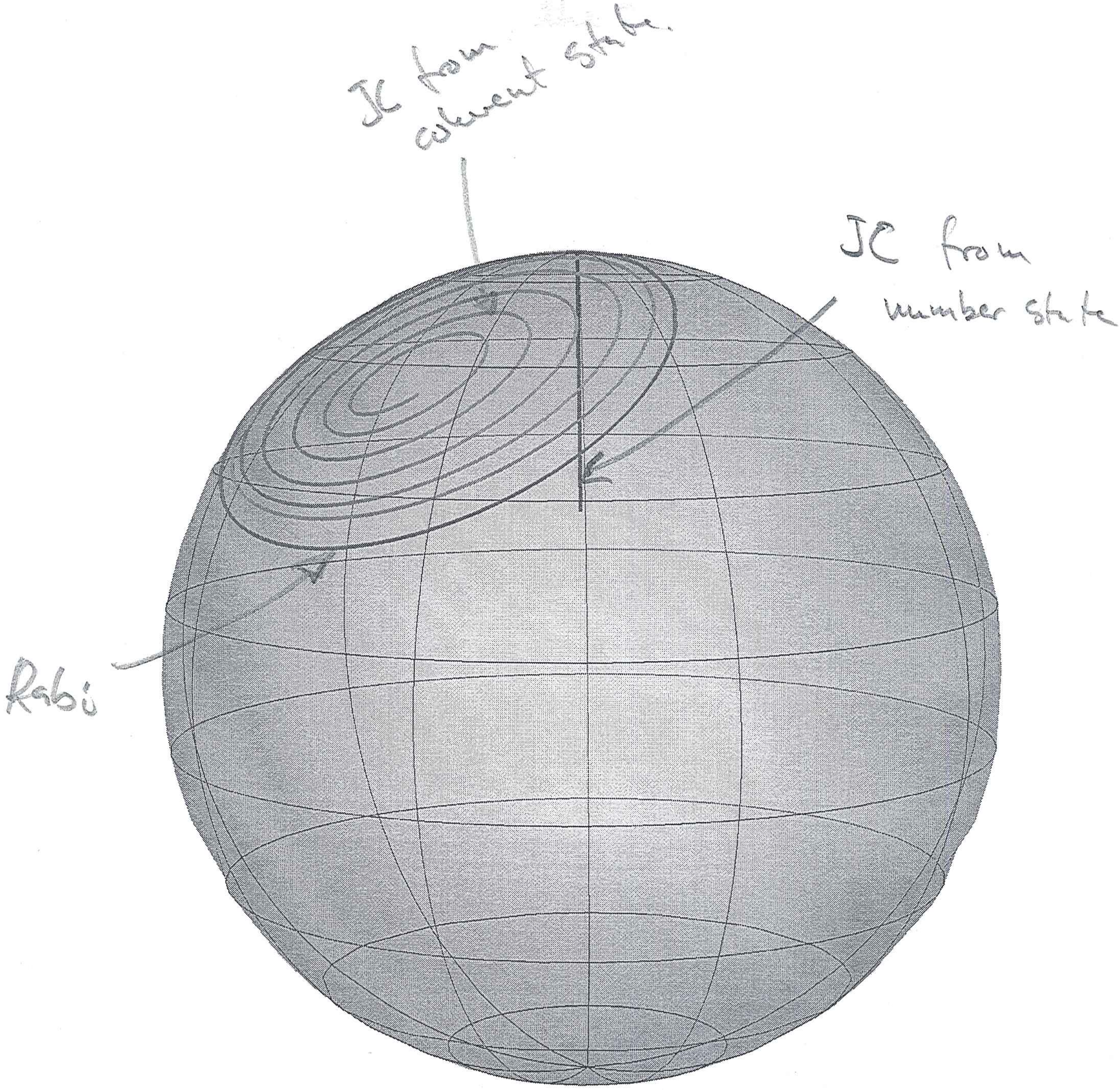

Jaynes-Cummings
Rabi

In the Rabi model, the state is always pure, and the Bloch vector presesses in a circle on the surface of the Bloch sphere. In the JC model, the qubit is entangled with the photon mode, and the reduced density matrix describes a mixed state. The Bloch vector oscillates along the axis of the Bloch sphere with $m_z^{JC} = m_z^R$.

e) $n \to \infty$: $\quad \Omega_n = \sqrt{\Delta^2 + g_n^2} = \sqrt{\Delta^2 + 4\lambda^2(n+1)} \to g_n$

$$\sin\theta_n = \frac{g_n}{\Omega_n} \to 1$$

The amplitude and frequency of the oscillations increase as $n \to \infty$, but the Bloch vector is always on the axis of the Bloch sphere and entanglement is not reduced. An idea for a classical limit is to assume that the photon mode starts in a coherent state instead of an eigenstate. We know that coherent states are the link to classical mechanics for the harmonic oscillator, and we can hope that it will extend to the JC model as well.



It works to some extent, but it becomes a spiral instead of circle. Here I used an average photon number of 9, maybe it should be bigger for the limit, but numerics gets slower. More work is needed...

Problem 2

a) $$H = \hbar\omega_r(a^\dagger a + \tfrac{1}{2}) + \frac{\hbar\Omega}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+)$$

$|\downarrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$
$|\uparrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Non-interacting eigenstates: $\{|\uparrow, n\rangle, |\downarrow, n\rangle\}$

We know that the interaction only mixes the states $|\downarrow, n\rangle$ and $|\uparrow, n+1\rangle$.

$$H|\downarrow, n\rangle = \underbrace{\left(\hbar\omega_r(n+\tfrac{1}{2}) + \frac{\hbar\Omega}{2}\right)}_{E_{\downarrow,n}}|\downarrow, n\rangle + \hbar g\sqrt{n+1}\,|\uparrow, n+1\rangle$$

$$H|\uparrow, n+1\rangle = \underbrace{\left(\hbar\omega_r(n+\tfrac{3}{2}) - \frac{\hbar\Omega}{2}\right)}_{E_{\uparrow,n+1}}|\uparrow, n+1\rangle + \hbar g\sqrt{n+1}\,|\downarrow, n\rangle$$

$\Delta = \Omega - \omega_r$

$$H_n = \frac{\hbar}{2}\begin{pmatrix}\Delta & 2g\sqrt{n+1}\\ 2g\sqrt{n+1} & -\Delta\end{pmatrix} + \hbar\omega_r(n+1)\,\mathbb{1}$$

$$= \frac{\hbar\Omega_n}{2}\begin{pmatrix}\cos\theta_n & \sin\theta_n\\ \sin\theta_n & -\cos\theta_n\end{pmatrix} + \hbar\omega_r(n+1)\,\mathbb{1}$$

$$= \frac{\hbar\Omega_n}{2}\big(\underbrace{\cos\theta_n\,\sigma_z + \sin\theta_n\,\sigma_x}_{\vec{n}\cdot\vec{\sigma}}\big) + \hbar\omega_r(n+1)\,\mathbb{1}$$

$\vec{n} = (\sin\theta_n, 0, \cos\theta_n)$

$\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$ $\quad \cos\theta_n = \dfrac{\Delta}{\Omega_n}$ $\quad \sin\theta_n = \dfrac{2g\sqrt{n+1}}{\Omega_n}$

$\vec{n}\cdot\vec{\sigma}$ has eigenvalues ± 1 and eigenstates

$$|+, n\rangle = \cos\theta_n|\downarrow, n\rangle + \sin\theta_n|\uparrow, n+1\rangle$$

$$|-, n\rangle = -\sin\theta_n|\downarrow, n\rangle + \cos\theta_n|\uparrow, n+1\rangle$$

These are also eigenstates of H_n and the eigenvalues are

$$E_{\pm n} = \pm\frac{\hbar\Omega_n}{2} + \hbar\omega_r(n+1)$$

b) For $\Delta \gg g$ the energies are

$$E_{\pm n} = \pm \frac{\hbar\Delta}{2}\sqrt{1+\frac{4g^2(n+1)}{\Delta^2}} + \hbar\omega_r(n+1)$$

$$\approx \pm \frac{\hbar\Delta}{2}\left(1+\frac{2g^2(n+1)}{\Delta^2}+\dots\right) + \hbar\omega_r(n+1)$$

$$= (n+1)\left(\hbar\omega_r \pm \frac{\hbar g^2}{\Delta}\right) \pm \frac{\hbar\Delta}{2}$$

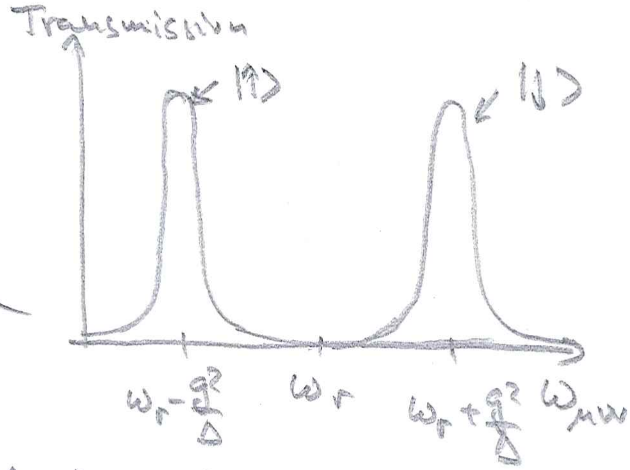
Level spacing: $E_{\pm,n+1} - E_{\pm n} = \hbar\omega_r \pm \frac{\hbar g^2}{\Delta}$ independent of n.

When $\Delta \gg g$ $\quad \cos\theta_n \approx 1 \quad \sin\theta_n \approx \frac{2g\sqrt{n+1}}{\Delta} \ll 1$

$\Rightarrow \quad |+,n\rangle \approx |\downarrow,n\rangle \qquad |-,n\rangle \approx |\uparrow,n\rangle$

$\Rightarrow$ Level spacing depends on qubit state.

c) The transmission is large when the microwave frequency ω_{mw} is resonant with transitions in the system. Since the level spacing depends on the qubit state we can determine it from the position of the resonance line.



The frequency should be choosen as one of the resonance frequencies, e.g. $\omega_r - \frac{g^2}{\Delta}$

If we get large transmission amplitude $\Rightarrow |\uparrow\rangle$

— " — small — " — $\Rightarrow |\downarrow\rangle$

d) We use $e^{\lambda A} B e^{-\lambda A} = B + \lambda [A,B] + \frac{\lambda^2}{2}[A,[A,B]] + \dots$

with $A = a\sigma^+ - a^\dagger \sigma^-$ and $B = H$.

Basic relations: $[a, a^\dagger] = 1 \qquad [\sigma^+, \sigma^-] = \sigma^z$

$$[\sigma^\pm, \sigma^z] = \mp 2\sigma^\pm$$

$$\sigma^+\sigma^- = \tfrac{1}{2}(1+\sigma^z)$$

$$\sigma^-\sigma^+ = \tfrac{1}{2}(1-\sigma^z)$$

$$[AB, C] = A[B,C] + [A,C]B$$

$$\sigma^+\sigma^- + \sigma^-\sigma^+ = 1$$

$$[a\sigma^+ - a^\dagger\sigma^-, a^\dagger a] = \underbrace{[a, a^\dagger a]}_{a^\dagger \underbrace{[a,a]}_{0} + \underbrace{[a,a^\dagger]}_{1} a}\sigma^+ - \underbrace{[a^\dagger, a^\dagger a]}_{a^\dagger \underbrace{[a^\dagger,a]}_{-1} + \underbrace{[a^\dagger,a^\dagger]}_{0} a}\sigma^- = a\sigma^+ + a^\dagger\sigma^-$$

$$[a\sigma^+ - a^\dagger\sigma^-, \sigma^z] = a\underbrace{[\sigma^+, \sigma^z]}_{-2\sigma^+} - a^\dagger \underbrace{[\sigma^-, \sigma^z]}_{2\sigma^-} = -2(a\sigma^+ + a^\dagger\sigma^-)$$

$$[a\sigma^+ - a^\dagger\sigma^-, a^\dagger\sigma^- + a\sigma^+] = \underbrace{[a\sigma^+, a^\dagger\sigma^-]}_{a\underbrace{[\sigma^+, a^\dagger\sigma^-]}_{a^\dagger\underbrace{[\sigma^+,\sigma^-]}_{\sigma^z}} + \underbrace{[a, a^\dagger\sigma^-]}_{\underbrace{[a,a^\dagger]}_{1}\sigma^-}\sigma^+} - \underbrace{[a^\dagger\sigma^-, a\sigma^+]}_{a^\dagger\underbrace{[\sigma^-, a\sigma^+]}_{-a\sigma^z} + \underbrace{[a^\dagger, a\sigma^+]}_{-\sigma^+}\sigma^-}$$

$$= \underbrace{aa^\dagger}_{a^\dagger a+1}\sigma^z + \sigma^-\sigma^+ + a^\dagger a\sigma^z + \sigma^+\sigma^-$$

$$= (2a^\dagger a + 1)\sigma^z + 1$$

$$[A, B] = -\hbar\Delta(a\sigma^+ + a^\dagger\sigma^-) + \hbar g[(2a^\dagger a+1)\sigma^z + 1]$$

$$[A,[A,B]] = -\hbar\Delta[(2a^\dagger a+1)\sigma^z + 1] + \underbrace{(\ )g}_{\text{Only contributes to } g^3}$$

$$U H U^\dagger \approx \hbar\omega_r\left(a^\dagger a+\tfrac{1}{2}\right)+\frac{\hbar\Omega}{2}\sigma^z+\hbar g\left(a^\dagger\sigma^-+a\sigma^+\right)$$
$$+\frac{g}{\Delta}\left[-\hbar\Delta\left(a\sigma^+ + a^\dagger\sigma^-\right)+\hbar g\left(\left(2a^\dagger a+1\right)\sigma^z+1\right)\right]$$
$$+\frac{1}{2}\left(\frac{g}{\Delta}\right)^2\left[-\hbar\Delta\right]\left[\left(2a^\dagger a+1\right)\sigma^z+1\right]$$
$$=\underbrace{\hbar\left(\omega_r+\frac{g^2}{\Delta}\sigma^z\right)a^\dagger a}_{\text{Level spacing.}}+\frac{\hbar}{2}\left(\Omega+\frac{g^2}{\Delta}\right)\sigma^z+\underbrace{\frac{1}{2}\hbar\omega_r+\frac{\hbar g^2}{2\Delta}}_{\text{Constant}}$$

$$\sigma^z|\downarrow\rangle = +|\downarrow\rangle \Rightarrow \hbar\omega_r+\frac{\hbar g^2}{\Delta}$$
$$\sigma^z|\uparrow\rangle = -|\uparrow\rangle \Rightarrow \hbar\omega_r-\frac{\hbar g^2}{\Delta}$$

as we found in b).

e) $$H_{\mu w}=\hbar\mathcal{E}\left(a^\dagger e^{-i\omega_{\mu w}t}+a e^{i\omega_{\mu w}t}\right)$$

$$\left[a\sigma^+-a^\dagger\sigma^-,\ a^\dagger e^{-i\omega_{\mu w}t}+a e^{i\omega_{\mu w}t}\right]=\sigma^+e^{-i\omega_{\mu w}t}+\sigma^-e^{i\omega_{\mu w}t}$$

$$U H_{\mu w}U^\dagger\approx\hbar\mathcal{E}\left(a^\dagger e^{-i\omega_{\mu w}t}+a e^{i\omega_{\mu w}t}\right)+\frac{\hbar g\mathcal{E}}{\Delta}\left(\sigma^+e^{-i\omega_{\mu w}t}+\sigma^-e^{i\omega_{\mu w}t}\right)$$

f) Transformation of Hamiltonian: $H'=THT^\dagger+i\hbar\frac{dT}{dt}T^\dagger$

$$T=e^{i\frac{\omega_{\mu w}t}{2}\sigma_z+i\omega_{\mu w}t\,a^\dagger a}$$

$$\frac{dT}{dt}T^\dagger=i\omega_{\mu w}\left(\frac{1}{2}\sigma_z+a^\dagger a\right)$$

T commutes with $UHU^\dagger$

$$e^{\lambda a^\dagger a} a e^{-\lambda a^\dagger a} = a + \lambda \underbrace{[a^\dagger a, a]}_{-a} + \frac{\lambda^2}{2!}\underbrace{[a^\dagger a,[a^\dagger a, a]]}_{a} + \cdots$$

$$= a - \lambda a + \frac{\lambda^2}{2!} a - \cdots = e^{-\lambda} a$$

$$\Rightarrow e^{i\omega_{\mu w} t\, a^\dagger a} a e^{-i\omega_{\mu w} t\, a^\dagger a} = a e^{-i\omega_{\mu w} t}$$

$$e^{i\omega_{\mu w} t\, a^\dagger a} a^\dagger e^{-i\omega_{\mu w} t\, a^\dagger a} = a^\dagger e^{i\omega_{\mu w} t}$$

$$e^{\lambda\sigma^z}\sigma^\pm e^{-\lambda\sigma^z} = \sigma^\pm + \lambda\underbrace{[\sigma^z,\sigma^\pm]}_{\pm 2\sigma^\pm} + \frac{\lambda^2}{2!}\underbrace{[\sigma^z,[\sigma^z,\sigma^\pm]]}_{4\sigma^\pm} + \cdots$$

$$= \sigma^\pm\left(1 \pm 2\lambda + \frac{(2\lambda)^2}{2!} \pm \frac{(2\lambda)^3}{3!} \cdots\right) = \sigma^\pm e^{\pm 2\lambda}$$

$$\Rightarrow \quad e^{i\frac{\omega_{\mu w}}{2} t\,\sigma^z}\sigma^\pm e^{-i\frac{\omega_{\mu w}}{2} t\,\sigma^z} = \sigma^\pm e^{\pm i\omega_{\mu w} t}$$

$$H_{iq} = TU(H + H_{\mu w})U^\dagger T^\dagger$$

$$= \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma^z\right) a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)\sigma^z$$

$$+ \hbar\varepsilon(a^\dagger + a) + \frac{\hbar\varepsilon g}{\Delta}\underbrace{(\sigma^+ + \sigma^-)}_{\sigma_x} - \hbar\omega_{\mu w}\left(\frac{1}{2}\sigma^z + a^\dagger a\right)$$

$$= \frac{\hbar}{2}\left[\Omega + 2\frac{g^2}{\Delta}\left(a^\dagger a + \frac{1}{2}\right) - \omega_{\mu w}\right]\sigma^z + \frac{\hbar\varepsilon g}{\Delta}\sigma_x$$

$$+ \hbar(\omega_r - \omega_{\mu w}) a^\dagger a + \hbar\varepsilon(a^\dagger + a)$$

g) With $\omega_{mw} = \Omega + (2n+1)\frac{g^2}{\Delta} - 2\frac{g\varepsilon}{\Delta}$ we have

$$\omega_r - \omega_{mw} = \underbrace{\omega_r - \Omega}_{-\Delta} + (\,)\frac{g}{\Delta} \approx -\Delta \quad \text{when } \Delta \gg g.$$

If we also assume $\Delta \gg \varepsilon$ the term $\hbar\varepsilon(a^\dagger a)$ will only induce small variations in the photon number n. We will ignore this term and replace $a^\dagger a \to n$.

$$H_{1q} = \frac{\hbar\varepsilon g}{\Delta}(\sigma^x + \sigma^z) + \text{constant}.$$

$$U(t) = e^{-\frac{i}{\hbar}H_{1q}t}$$

$$U\left(\frac{\pi\Delta}{2\sqrt{2}g\varepsilon}\right) = e^{-\frac{i\pi}{2}\frac{1}{\sqrt{2}}(\sigma^x+\sigma^z)} = \underbrace{\cos\frac{\pi}{2}}_{0}\cdot\mathbb{1} - i\underbrace{\sin\frac{\pi}{2}}_{1}\frac{1}{\sqrt{2}}(\sigma^x+\sigma^z)$$

$$= -\frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -i\cdot H$$

h) Let $\omega_{mw} = \Omega + \frac{g^2}{\Delta}(2n+1) \Rightarrow H_{1q} = \frac{\hbar\varepsilon g}{\Delta}\sigma_x + \text{constant}$

Rotation around x-axis with angle θ: $e^{-i\frac{\theta}{2}\sigma_x}$

$$\Rightarrow \frac{\varepsilon g}{\Delta}t = \frac{\theta}{2} \quad \Rightarrow t = \frac{\Delta\theta}{2g\varepsilon}$$

i) $H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) + \frac{\hbar\Omega}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$ (13)

$$U = e^{\frac{g}{\Delta}[\underbrace{a(\sigma_1^+ + \sigma_2^+) - a^\dagger(\sigma_1^- + \sigma_2^-)}_{A}]}$$

$$[A, a^\dagger a] = a(\sigma_1^+ + \sigma_2^+) + a^\dagger(\sigma_1^- + \sigma_2^-)$$

$$[A, \sigma_1^z + \sigma_2^z] = -2[a(\sigma_1^+ + \sigma_2^+) + a^\dagger(\sigma_1^- + \sigma_2^-)]$$

$$[A, a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$$

$$= [a(\sigma_1^+ + \sigma_2^+), a^\dagger(\sigma_1^- + \sigma_2^-)] - [a^\dagger(\sigma_1^- + \sigma_2^-), a(\sigma_1^+ + \sigma_2^+)]$$

$$= 2[\underbrace{a a^\dagger}_{a^\dagger a + 1}(\sigma_1^z + \sigma_2^z) + \underbrace{(\sigma_1^- + \sigma_2^-)(\sigma_1^+ + \sigma_2^+)}_{1 - \frac{1}{2}(\sigma_1^z + \sigma_2^z) + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+}]$$

$$= 2[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+]$$

$$[A, H] = -\hbar\Delta[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + 2\hbar g[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+]$$

$$[A, [A, H]] = -\hbar\Delta\, 2[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+] + (\;)\, g$$

$$UHU^\dagger = \hbar\omega_r(a^\dagger a + \tfrac{1}{2}) + \frac{\hbar\Omega}{2}(\sigma_1^z + \sigma_2^z) + \hbar g[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)]$$

$$+ \frac{g}{\Delta}\{-\hbar\Delta[a^\dagger(\sigma_1^- + \sigma_2^-) + a(\sigma_1^+ + \sigma_2^+)] + 2\hbar g[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+]\}$$

$$+ \frac{1}{2}\left(\frac{g}{\Delta}\right)^2(-2\hbar\Delta)[(a^\dagger a + \tfrac{1}{2})(\sigma_1^z + \sigma_2^z) + 1 + \sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+]$$

$$= \hbar\left[\omega_r + \frac{g^2}{\Delta}(\sigma_1^z + \sigma_2^z)\right]a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)(\sigma_1^z + \sigma_2^z)$$

$$+ \frac{\hbar g^2}{\Delta}(\sigma_1^- \sigma_2^+ + \sigma_2^- \sigma_1^+) + \text{constant}$$

j) Transformation to rotating frame

$$T(t) = e^{i\frac{\Omega t}{2}(\sigma_1^z+\sigma_2^z) + i\omega_r t a^\dagger a} \qquad \frac{dT}{dt}T^\dagger = i\frac{\Omega}{2}(\sigma_1^z+\sigma_2^z) + i\omega_r a^\dagger a$$

$T(t)$ commutes trivially with the two first terms in $UHU^\dagger$, and in fact also

$$[\sigma_1^z+\sigma_2^z, \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+] = [\sigma_1^z,\sigma_1^-]\sigma_2^+ + [\sigma_1^z,\sigma_1^+]\sigma_2^- + [\sigma_2^z,\sigma_2^-]\sigma_1^+ + [\sigma_2^z,\sigma_2^+]\sigma_1^-$$

$$= -2\sigma_1^-\sigma_2^+ + 2\sigma_1^+\sigma_2^- - 2\sigma_2^-\sigma_1^+ + 2\sigma_2^+\sigma_1^- = 0$$

So
$$H_{2q} = TUHU^\dagger T^\dagger + i\hbar\frac{dT}{dt}T^\dagger$$
$$= \frac{\hbar g^2}{\Delta}(\sigma_1^z+\sigma_2^z)(a^\dagger a+\tfrac{1}{2}) + \frac{\hbar g^2}{\Delta}(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)$$

k) We have shown that the to terms in H_{2q} commute

$$\Rightarrow U_{2q} = e^{-\frac{i}{\hbar}H_{2q}t} = e^{-\frac{ig^2}{\Delta}t(\sigma_1^z+\sigma_2^z)(a^\dagger a+\frac{1}{2})}\, e^{-\frac{ig^2}{\Delta}t\underbrace{(\sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+)}_{A}}$$

$$A = \sigma_1^-\sigma_2^+ + \sigma_2^-\sigma_1^+ = \begin{pmatrix}0&0\\1&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&0&0&0\\0&\multicolumn{2}{c}{\multirow{2}{*}{(σ_x)}}&0\\0&&&0\\0&0&0&0\end{pmatrix}$$

$$e^{-i\frac{g^2t}{\Delta}A} = \begin{pmatrix}1&0&0&0\\0&\multicolumn{2}{c}{\multirow{2}{*}{$\left(e^{-\frac{ig^2}{\Delta}t\sigma_x}\right)$}}&0\\0&&&0\\0&0&0&1\end{pmatrix}$$

$$e^{-\frac{ig^2}{\Delta}t\sigma_x} = \cos\frac{g^2t}{\Delta}\cdot 1 - i\sin\frac{g^2t}{\Delta}\cdot\sigma_x = \begin{pmatrix}\cos\frac{g^2t}{\Delta} & -i\sin\frac{g^2t}{\Delta}\\ -i\sin\frac{g^2t}{\Delta} & \cos\frac{g^2t}{\Delta}\end{pmatrix}$$

1.) $t = \frac{3\pi\Delta}{2g^2} \Rightarrow \frac{g^2 t}{\Delta} = \frac{3\pi}{2} \Rightarrow M\left(\frac{3\pi\Delta}{2g^2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

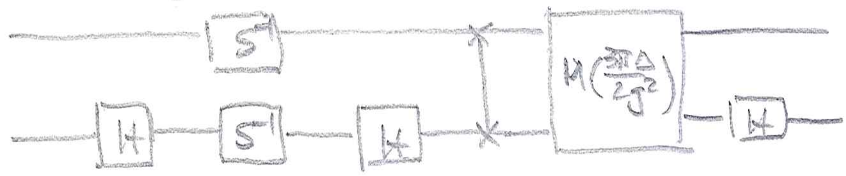
A Hadamard gate on the second qubit is

$$\mathbb{1} \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

S^{-1} on both qubits is

$$S^{-1} \otimes S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The circuit



is then given by

$$\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\mathbb{1} \otimes H} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{M\left(\frac{3\pi\Delta}{2g^2}\right)} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{SWAP} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{S^{-1} \otimes S^{-1}} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\mathbb{1} \otimes H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = CNOT$$