Problem set 12

12.1 A state in thermal equilibrium (Exam 2011)

A quantum state in thermal equilibrium is described by the density operator

$$\hat{\rho}(\beta) = N(\beta)e^{-\beta\hat{H}} = N(\beta)\sum_n e^{-\beta E_n}|n\rangle\langle n|$$

with $\hat{H}$ as the Hamiltonian, E_n as the corresponding energy eigenvalues, and $N(\beta)$ as a normalization factor. The parameter β is related to the temperature T by $\beta = 1/(k_B T)$, with k_B as Boltzmanns constant.

a) Show that the expectation value for the energy can be expressed in terms of $N(\beta)$ as

$$E(\beta) = \frac{d}{d\beta}\ln N(\beta)$$

and find a similar expression for the von Neumann entropy $S(\beta) = \mathrm{Tr}[\hat{\rho}(\beta)\ln\hat{\rho}(\beta)]$. (Use here the natural logarithm in the definition of S.)

b) For a two-level system, with Hamiltonian $\hat{H} = (\epsilon/2)\sigma_z$, determine the functions $N(\beta)$, $E(\beta)$ and $S(\beta)$, and make a sketch of the expectation value of the energy E as function of the temperature T.

c) Find the density operator expressed in the form $\hat{\rho} = (1/2)(\mathbb{1} + \mathbf{r}\cdot\sigma)$. Determine $\mathbf{r}$ as a function of β and relate this to the results in b).

12.2 Coupled harmonic oscillators (Exam 2016)

Two harmonic oscillators, referred to as A and B, form a composite quantum mechanical system. The Hamiltonian of the system has the form

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + \mathbb{1}) + \hbar\lambda(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a})$$

with $(\hat{a}, \hat{a}^\dagger)$ as lowering and raising operators for A and $(\hat{b}, \hat{b}^\dagger)$ as corresponding operators for B, while ω and λ are real valued constants.

a) Show that the Hamilton operator can be expressed in diagonal form as

$$\hat{H} = \hbar\omega_c\hat{c}^\dagger\hat{c} + \hbar\omega_d\hat{d}^\dagger\hat{d} + \hbar\omega\mathbb{1}$$

where $\hat{c}$ and $\hat{d}$ are linear combinations of $\hat{a}$ and $\hat{b}$,

$$\hat{c} = \mu\hat{a} + \nu\hat{b}, \qquad \hat{d} = -\nu\hat{a} + \mu\hat{b},$$

with μ and ν as real constants satisfying $\mu^2 + \nu^2 = 1$, and determine the new parameters μ, ν, ω_c and ω_d, expressed in terms of ω and λ. Check that the new operators $\hat{c}$ and $\hat{d}$ satisfy the same set of harmonic oscillator commutation relations as $\hat{a}$ and $\hat{b}$. It is sufficient to show

$$\left[\hat{c}, \hat{c}^\dagger\right] = \left[\hat{d}, \hat{d}^\dagger\right] = 1, \qquad \left[\hat{c}, \hat{d}^\dagger\right] = 0$$

b) Assume that the state $|\psi(0)\rangle$ of the composite system, at time $t = 0$, is a coherent state when expressed in terms of the new variables,

$$\hat{c}|\psi(0)\rangle = z_{c0}|\psi(0)\rangle, \qquad \hat{d}|\psi(0)\rangle = z_{d0}|\psi(0)\rangle.$$

Also at a later time the state $|\psi(t)\rangle$ will be a coherent state for both $\hat{c}$ and $\hat{d}$,

$$z_c(t) = e^{i\omega_c t} z_{c0}, \qquad z_d(t) = e^{i\omega_d t} z_{d0}.$$

Show this for $z_c(t)$. (The expression for $z_d(t)$ follows in the same way, and is therefore not needed to be shown.)

c) Show that the state $|\psi(t)\rangle$ is a coherent state also for the original harmonic oscillator operators $\hat{a}$ and $\hat{b}$, and find the eigenvalues $z_a(t)$ and $z_b(t)$ expressed in terms of z_{a0} and z_{b0} .

12.3 Time evolution in a two-level system (Exam 2013)

The Hamiltonian of a two-level system (denoted A) is $\hat{H}_0 = (1/2)\hbar\omega\,\sigma_z$, with σ_z as the diagonal Pauli matrix. We refer to the normalized ground state vector as $|g\rangle$ and the exited state as $|e\rangle$. In reality the system is coupled to a radiation field (denoted S), and the excited state will therefore decay to the ground state under emission of a quantum of radiation. $\hat{\rho}$ denotes the reduced density operator of subsystem A. To a good approximation the time evolution of this system is described by the Lindblad equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\,[H_0, \hat{\rho}] - \frac{1}{2}\gamma \left[\hat{\alpha}^\dagger\hat{\alpha}\hat{\rho} + \hat{\rho}\hat{\alpha}^\dagger\hat{\alpha} - 2\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger\right] \tag{1}$$

with γ as the decay rate for the transition $|e\rangle \to |g\rangle$, $\hat{\alpha} = |g\rangle\langle e|$ and $\hat{\alpha}^\dagger = |e\rangle\langle g|$.

In matrix form, with $\{|e\rangle, |g\rangle\}$ as basis, we write the density matrix as $\hat{\rho}$

$$\hat{\rho} = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{2}$$

with p_e as the probability for the system to be in state $|e\rangle$ and p_g as the probability for the system to be in state $|g\rangle$.

a) Assume initially the two-level system, at time $t = 0$, to be in state $\hat{\rho} = |e\rangle\langle e|$. Show, by use of Eq. (1), that p_e decays exponentially, with γ as decay rate, while the total probability $p_e + p_g$ is conserved.

b) Assume next that the system is initially in the following superposition of the two eigenstates of $\hat{H}_0$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Determine the time dependent density matrix $\hat{\rho}(t)$ with this initial state.

c) The density operator of subsystem A can alternatively be expressed in terms of the Pauli matrices as $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$. Determine the function $r^2(t)$ in the two cases above and show that in both cases it has a minimum for $t = (1/\gamma)\ln 2$. What is the minimum value for r in the two cases? Comment on the implication the results give for the entanglement between the two subsystems A and S. (We assume A+S all the time to be in a pure state.)