Problem set 3

3.1 Spin operators and Pauli matrices

a) We set $\mathbf{n} = (n_1, n_2, n_3)$, then

$$\mathbf{n} \cdot \sigma = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 = \begin{pmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{pmatrix} \tag{1}$$

The eigenvalue equation is:

$$\det\left(\sigma_{\mathbf{n}} - \mathbb{1}\right) = 0 \quad \Rightarrow \quad \det \begin{pmatrix} n_3 - \lambda & n_1 - in_2 \\ n_1 + in_2 & -n_3 - \lambda \end{pmatrix} = 0$$

$$\begin{aligned} -\left(n_3 - \lambda\right)\left(n_3 + \lambda\right) - \left|n_1 + in_2\right|^2 &= 0 \\ \Rightarrow -n_3^2 + \lambda^2 - \left(n_1^2 + n_2^2\right) &= 0 \\ \Rightarrow \lambda^2 &= n_1^2 + n_2^2 + n_3^2 \\ \Rightarrow \lambda &= \pm\left|\mathbf{n}\right|^2 = \pm 1 \end{aligned}$$

The corresponding eigenstate equation for $\lambda = 1$ is:

$$\sigma_{\mathbf{n}}\Psi_{\mathbf{n}} = \Psi_{\mathbf{n}}$$

By switching to spherical coordinates, we get $\mathbf{n} = (n_1, n_2, n_3) = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. Inserting this into the matrix (1), the equation gets the form:

$$\begin{pmatrix} \cos\theta & (\cos\phi - i\sin\phi)\sin\theta \\ (\cos\phi + i\sin\phi)\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Which corresponds to these equations:

$$\begin{aligned} \psi_1\cos\theta + \psi_2 e^{-i\phi}\sin\theta &= \psi_1 \\ \psi_1 e^{i\phi}\sin\theta - \psi_2\cos\theta &= \psi_2 \end{aligned}$$

We need to solve one of them, I'm choosing the second, where I get the relation:

$$\frac{e^{i\phi}\sin\theta}{1 + \cos\theta}\psi_1 = \psi_2$$

Then:

$$\frac{e^{i\phi}\sin\theta}{1 + \cos\theta} = \frac{e^{i\phi}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}} = \frac{e^{i\phi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

This gives us the relation:

$$e^{i\phi}\sin\frac{\theta}{2}\psi_1 = \psi_2\cos\frac{\theta}{2}$$

This is true for $\psi_1 = \cos\frac{\theta}{2}$ and $\psi_2 = e^{i\phi}\sin\frac{\theta}{2}$, which defines the eigenvectors for $\lambda = 1$:

$$\Psi_{\mathbf{n}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

For $\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \mathbf{n}$, we get:

$$\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \left(\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}}\right)$$

$$\begin{aligned}
\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}} &= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} e^{i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \\
&= e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\
&= \frac{1}{2}\sin\theta\left(e^{i\phi} + e^{-i\phi}\right) \\
&= \cos\phi\sin\theta = n_1
\end{aligned}$$

$$\begin{aligned}
\Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}} &= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} -ie^{i\phi}\sin\frac{\theta}{2} \\ i\cos\frac{\theta}{2} \end{pmatrix} \\
&= i\left(-e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\
&= \frac{1}{2}i\sin\theta\left(e^{-i\phi} - e^{i\phi}\right) \\
&= -i^2\sin\theta\sin\phi = n_2
\end{aligned}$$

$$\begin{aligned}
\Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}} &= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \left(\cos\tfrac{\theta}{2}, \quad e^{-i\phi}\sin\tfrac{\theta}{2}\right)\begin{pmatrix} \cos\frac{\theta}{2} \\ -e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \\
&= \cos^2\frac{\theta}{2} - e^{-i\phi}e^{i\phi}\sin^2\frac{\theta}{2} \\
&= \cos\theta = n_3
\end{aligned}$$

Thus:

$$\Psi_{\mathbf{n}}^\dagger \sigma \Psi_{\mathbf{n}} = \left(\Psi_{\mathbf{n}}^\dagger \sigma_1 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_2 \Psi_{\mathbf{n}}, \Psi_{\mathbf{n}}^\dagger \sigma_3 \Psi_{\mathbf{n}}\right) = (n_1, n_2, n_3) = \mathbf{n}$$

b)

$$e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z}$$

From problem set 2, we have:

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}} = \hat{B} + \lambda\left[\hat{A},\hat{B}\right] + \frac{\lambda^2}{2}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \cdots \tag{2}$$

We choose $\hat{A} = -\frac{i}{2}\sigma_z$, we get our expression on the same form as in (2):

$$\begin{aligned} e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} &= \sigma_x + \alpha\left[-\frac{i}{2}\sigma_z,\sigma_x\right] + \frac{\alpha^2}{2}\left[-\frac{i}{2}\sigma_z,\left[-\frac{i}{2}\sigma_z,\sigma_x\right]\right] + \frac{\alpha^3}{6}\left[-\frac{i}{2}\sigma_z,\left[-\frac{i}{2}\sigma_z,\left[-\frac{i}{2}\sigma_z,\sigma_x\right]\right]\right] + \cdots \\ &= \sigma_x + \alpha\left(-\frac{i}{2}\right)\left[\sigma_z,\sigma_x\right] + \frac{\alpha^2}{2}\left(-\frac{i}{2}\right)^2\left[\sigma_z,\left[\sigma_z,\sigma_x\right]\right] + \frac{\alpha^3}{6}\left(-\frac{i}{2}\right)^3\left[\sigma_z,\left[\sigma_z,\left[\sigma_z,\sigma_x\right]\right]\right] + \cdots \end{aligned}$$

Then we know that the commutators are such that $[\sigma_x,\sigma_y] = 2i\epsilon_{xyz}\sigma_z$ where $\epsilon_{xyz} = 1$ and any odd number of permutations returns -1, and any even number of permutations return 1. Thus: $[\sigma_z,\sigma_x] = 2i\sigma_y$, $[\sigma_z,\sigma_y] = -2i\sigma_x$, and

$$\begin{aligned} e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} &= \sigma_x + \alpha\left(-\frac{i}{2}\right)2i\sigma_y + \frac{\alpha^2}{2}\left(-\frac{i}{2}\right)^2(2i)\left[\sigma_z,\sigma_y\right] \\ &\quad + \frac{\alpha^3}{6}\left(-\frac{i}{2}\right)^3(2i)\left[\sigma_z,\left[\sigma_z,\sigma_y\right]\right] + \cdots \qquad (3) \\ &= \sigma_x + \alpha\left(-\frac{i}{2}\right)2i\sigma_y - \frac{\alpha^2}{2}\left(-\frac{i}{2}\right)^2(2i)^2\sigma_x - \frac{\alpha^3}{6}\left(-\frac{i}{2}\right)^3(2i)^2\left[\sigma_z,\sigma_x\right] + \cdots \\ &= \sigma_x + \alpha\sigma_y - \frac{\alpha^2}{2}\sigma_x - \frac{\alpha^3}{6}\sigma_y + \cdots \qquad (4) \\ &= \left(1 - \frac{\alpha^2}{2} + \cdots\right)\sigma_x + \left(\alpha - \frac{\alpha^3}{6} + \cdots\right)\sigma_y \end{aligned}$$

The series continues in the familiar pattern of $\cos\alpha$ and $\sin\alpha$ due to the pattern in (3), which leaves us:

$$e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} = \cos\alpha\sigma_x + \sin\alpha\sigma_y \tag{5}$$

The unitary matrix $\hat{U} = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}$, when transforming an operator/matrix, causes the transformations:

$$\sigma \rightarrow \hat{U}\sigma\hat{U}^\dagger = e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}\sigma e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}$$

As we saw in (5), if $\mathbf{n} = (0,0,1) = z$, we got a rotation about the z axis. If we now imagine a different orthonormal coordinate system with unit vectors $\mathbf{n} = \mathbf{n}_z, \mathbf{n}_y, \mathbf{n}_x$, then the transformation $\hat{U}\sigma_{\mathbf{n}_x}\hat{U}^\dagger$ would look like:

$$\sigma_{\mathbf{n}_x} \rightarrow e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}}\sigma e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \cos\alpha\sigma_{\mathbf{n}_x} + \sin\alpha\sigma_{\mathbf{n}_y}$$

and rotates the spin basis around the axis $\mathbf{n}$.

c)

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \sum_{k=0}^{\infty}\frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^k}{k!}$$

Then we need to know what the different powers of $\sigma_{\mathbf{n}}$ are:

$$\begin{aligned}
\sigma_{\mathbf{n}}^2 &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}^2 = \begin{pmatrix} \cos^2\theta + e^{-i\phi}e^{i\phi}\sin^2\theta & \cos\theta e^{-i\phi}\sin\theta - \cos\theta e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta\cos\theta - \cos\theta e^{i\phi}\sin\theta & e^{i\phi}\sin\theta e^{-i\phi}\sin\theta + \cos^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \mathbb{1}
\end{aligned}$$

Thus, we see that $\sigma_{\mathbf{n}}$ follows the standard Pauli matrix identites: $\sigma_{\mathbf{n}}^{2k} = \mathbb{1}$ and $\sigma_{\mathbf{n}}^{2k+1} = \sigma_{\mathbf{n}}$ for $k = 0, 1, \ldots$ This let's us rewrite our series as:

$$e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} = \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^k}{k!} = \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{\left(-\frac{i}{2}\alpha\sigma_{\mathbf{n}}\right)^{2k+1}}{(2k+1)!}$$

Using that $(-i)^{2k} = (-1)^k$, $(-i)^{2k+1} = -i^{2k+1} = -i^{2k}i = (-1)^{k+1}\,i$, and the identities for the Pauli matrices, we get:

$$\begin{aligned}
e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} &= \underbrace{\mathbb{1}\sum_{k=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{2k}(-1)^k}{(2k)!}}_{=\cos\frac{\alpha}{2}} + \underbrace{i\sigma_{\mathbf{n}}\sum_{k=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{2k+1}(-1)^{k+1}}{(2k+1)!}}_{=\sin\frac{\alpha}{2}} \\
e^{-\frac{i}{2}\alpha\sigma_{\mathbf{n}}} &= \cos\frac{\alpha}{2}\mathbb{1} + i\sin\frac{\alpha}{2}\sigma_{\mathbf{n}}
\end{aligned}$$