Problem set 4

4.1 Ladder operators in the Heisenberg picture

Consider a harmonic oscillator with Hamiltonian

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{1}$$

expressed in terms of the ladder operators $\hat{a}^\dagger$ and $\hat{a}$. Show that these two operators take the following time dependent form in the Heisenberg picture

$$\hat{a}^\dagger(t) = e^{i\omega t}\hat{a}^\dagger\,, \quad \hat{a}(t) = e^{-i\omega t}\hat{a} \tag{2}$$

4.2 Displacement operators in phase space

For a particle moving in one dimension the position coordinate x and the momentum p define the coordinates of the two-dimensional classical *phase space*.

In the quantum description of the one-dimensional harmonic oscillator non-hermitian lowering operator is defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\,\hat{x} + i\hat{p}) \tag{3}$$

We may consider this as the operator of a complex phase space variable, with position as the real part and momentum as the imaginary part. It has a dimensionless form due to the constants introduced in the expression.

A coherent state, in a similar way is characterized by a complex number z, the eigenvalue of $\hat{a}$, which we may interpret as a complex phase space coordinate,

$$z = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\,x_c + ip_c) \tag{4}$$

The following operator

$$\hat{\mathcal{D}}(z) = e^{(z\hat{a}^\dagger - z^*\hat{a})} \tag{5}$$

acts as a displacement operator in phase space, in the sense

$$\hat{\mathcal{D}}(z)^\dagger\hat{x}\hat{\mathcal{D}}(z) = \hat{x} + x_c\,, \quad \hat{\mathcal{D}}(z)^\dagger\hat{p}\hat{\mathcal{D}}(z) = \hat{p} + p_c \tag{6}$$

Show that displacements in two different directions in general will not commute but rather satisfy a relation of the form

$$\hat{\mathcal{D}}(z_a)\hat{\mathcal{D}}(z_b) = e^{i\alpha(z_a,z_b)}\hat{\mathcal{D}}(z_b)\hat{\mathcal{D}}(z_a) \tag{7}$$

with $\alpha(z_a, z_b)$ as a complex phase. Determine the phase as a function of z_a and z_b. What is the condition for the two operators to commute?

4.3 Eigenvectors for $\hat{a}^\dagger$?

The coherent states $|z\rangle$ are defined as eigenvectors of the lowering operator $\hat{a}$. Assume $|\bar{z}\rangle$ to be eigenvector of the raising operator $\hat{a}^\dagger$,

$$\hat{a}^\dagger|\bar{z}\rangle = \bar{z}|\bar{z}\rangle \tag{8}$$

Show that no normalizable vector exists that satisfies this equation by expanding the state $|\bar{z}\rangle$ in the energy eigenstates $|n\rangle$.

4.4 A driven harmonic oscillator (Exam 2010)

A quantum mechanical, driven harmonic oscillator is described by the following Hamiltonian

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar\lambda(\hat{a}^\dagger e^{-i\omega t} + \hat{a}e^{i\omega t}) \tag{9}$$

where $\hat{a}$ og $\hat{a}^\dagger$ satisfy the standard commutation relations for lowering and raising operators, and where ω_0, ω og λ are three constants. We introduce the following dimensionless position and momentum operators,

$$\hat{x} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)\,, \quad \hat{p} = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger) \tag{10}$$

a) As a reminder, Heisenberg's equation of motion has the form

$$\frac{d}{dt}\hat{A} = \frac{i}{\hbar}\left[H, \hat{A}\right] + \frac{\partial}{\partial t}\hat{A} \tag{11}$$

for any given observable $\hat{A}$. Apply this to the lowering operator $\hat{a}$, and show that it satisfies an equation of the form

$$\frac{d^2\hat{x}}{dt^2} + \omega_0^2\hat{x} = C\cos\omega t \tag{12}$$

Determine the constant C.

b) By use of the the time dependent unitary transformation

$$\hat{T}(t) = e^{i\omega t\,\hat{a}^\dagger\hat{a}} \tag{13}$$

the new Hamiltonian, $\hat{H}_T(t)$, which determines the time evolution of the transformed state vectors $|\psi_T(t)\rangle = \hat{T}(t)|\psi(t)\rangle$, will take a time independent form. Find the expression for the transformed Hamiltonian.

c) A coherent state is defined as an eigenstate of the lowering operator $\hat{a}$,

$$\hat{a}|z\rangle = z|z\rangle \tag{14}$$

Assume at time $t = 0$ the oscillator is in the ground state for the λ-independent part of the Hamiltonian, that is

$$|\psi(0)\rangle = |0\rangle\,, \qquad \hat{a}|0\rangle = 0 \tag{15}$$

Show that, during the time evolution, it will continue as a coherent state,

$$|\psi(t)\rangle = e^{i\alpha(t)}|z(t)\rangle \tag{16}$$

with $\alpha(t)$ as a time dependent phase and $z(t)$ as a complex-valued function of time.

Find the function $z(t)$ and give a qualitative description of the motion in the complex z-plane. Show that the real part $x(t) = (z(t) + z(t)^*)/2$ satisfies the same equation of motion (12) as the position operator $\hat{x}(t)$.

As a reminder we give the following operator relation

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \left[\hat{A}, \hat{B}\right] + \frac{1}{2!}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + ... \tag{17}$$

which applies generally to two operators $\hat{A}$ og$\hat{B}$.