# Problem set 6

### 6.1 Entanglement

Two persons A and B communicate with the help of quantum entanglement. They share a set of pairs of particles with spins in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle) \tag{1}$$

where $|++\rangle = |+\rangle \otimes |+\rangle$ is a state where both particles of the pair have *spin up* in the $z$-direction, and similarly $|--\rangle = |-\rangle \otimes |-\rangle$ is the state where both particles have *spin down* in the $z$-direction.

a) What is the quantity used as measure for the degree of entanglement in such a two-partite system, and what is the degree of entanglement in the given spin state?

b) Assume A and B perform independent spin operations on their particles in a given pair, each operation described by a unitary operator, $\hat{U}_A$ or $\hat{U}_B$. What happens to the entanglement of the two-particle system under such an operation.

c) Assume A performs an ideal measurement of the spin component in the $x$- direction, which projects the spin to an eigenstate of the x-component of the spin operator. What happens to the entanglement in this case?

### 6.2 Schmidt decomposition 1

We have a system consisting of two spin-$\frac{1}{2}$ particles. For each of the following states, study the reduced density matrix of of one of the particles and determine if the state is entangled or not. For the states which are not entangled, find a factorization of the state as a tensor product of one state for each particle. For the entagled states, find the Schmidt decomposition of the state.

$$|\psi_1\rangle = \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$$
$$|\psi_2\rangle = \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$$
$$|\psi_3\rangle = a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle$$
$$|\psi_4\rangle = a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle$$

where

$$a_\pm = \frac{\sqrt{3} \pm 1}{4}$$

### 6.3 Schmidt decomposition 2

Entanglement can occur not only between distinct particles, but also between different observables fot the same particle, like position and spin. Here we will find the Schmidt decomposition of one continuos and one discrete Hilbert space. A spin-half particle moving in one dimension is described by a two-component wave function

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{2}$$

where the upper matrix position is assumed to correspond to "spin up" in the $z$-direction and the lower matrix position to "spin down" in the same direction. The scalar product of the two wave functions will generally be different from zero, and we write it as

$$\langle \psi_1 | \psi_2 \rangle = \int dx \, \psi_1^*(x) \psi_2(x) \equiv \Delta \tag{3}$$

a) The Schmidt decomposition of the two-component wave function has the form

$$\Psi(x) = c_1 \chi_1 \, \phi_1(x) + c_2 \chi_2 \, \phi_2(x) \tag{4}$$

where $c_1$ and $c_2$ are expansion coefficients, $\chi_1$ and $\chi_2$ are normalized, two-component spinors, and $\phi_1(x)$ and $\phi_2(x)$ are normalized, scalar (one-component) wave functions. What are the conditions that the spinors and wave functions should satisfy?

b) Assume the two wave functions of (2) are real Gaussian functions of the form

$$\psi_1(x) = N e^{-\lambda(x-x_0)^2} , \quad \psi_2(x) = N e^{-\lambda(x+x_0)^2} \tag{5}$$

Determine the normalization factor $N$ and the overlap $\Delta$, expressed in terms of $\lambda$ and $x_0$.

c) Determine the coefficients, spinors and wave functions in (4). (Since the wave function $\Psi(x)$ is real, you may assume the variables in Eq.(4) all to be real.)

## 6.4 Coupled two-level systems

Two coupled two-level systems $A$ and $B$ are described by the following Hamiltonian

$$\hat{H} = \frac{\epsilon}{2}(3\,\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z) + \lambda(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+) \tag{6}$$

where the first factor in the tensor product refers to system $A$ and the second factor to system $B$. In the equation we use the definition $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$.

a) Write the Hamiltonian as a 4x4 matrix and show that two of the eigenvalues and eigenvectors are independent of $\lambda$. Introduce new variables, defined by $\epsilon = \mu \cos\theta$ and $\lambda = \mu \sin\theta$. Solve the eigenvalue problem for the remaining two-dimensional subspace and determine both the energies and eigenvectors as functions of $\mu$ and $\theta$.

b) Express the two eigenstates as 4x4 density matrices and determine the reduced density matrices for the two subsystems $A$ and $B$.

c) Determine the entropy of the reduced density matrices as functions of $\theta$. For what parameter value is the entanglement of the two subsystems maximal?

### 6.5 Entanglement and measurements

In a textbook on quantum mechanics we find the following discussion of the EPR thought experiment:

> The problem posed by Einstein, Rosen, and Podolsky was made sharper by David Bohm (1917–1992). A system of zero total angular momentum decays into two particles, each with spin 1/2. Using the Clebsch-Gordan coefficients for combining spin 1/2 and spin 1/2 to make spin zero, the spin state vector is then
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> $$\Psi = \frac{1}{\sqrt{2}} \left[ \Psi_{\uparrow\downarrow} - \Psi_{\downarrow\uparrow} \right], \tag{12.1.2}$$
>
> where the two arrows indicate the signs of the $z$-component of the two particles spins. After a long time, the particles are far apart, and then measurements are made of the spin components of particle 1. If the $z$-component of the spin of particle 1 is measured, it must have a value $\hbar/2$ or $\hbar/2$, and then the $z$-component of the spin of particle 2 must correspondingly have a value $\hbar/2$ or $+\hbar/2$, respectively. This not mysterious – the particles were once in contact, so it is not surprising that the $z$-components of their spins are strongly correlated. Following this measurement, suppose that the $x$-component of the spin of particle 1 is measured. It will be found to have the value $\hbar/2$ or $-\hbar/2$, and the $z$-component of particle 1s spin will no longer have a definite value. Also, because the system has zero total angular momentum, the spin of particle 2 will then have $x$-component $-\hbar/2$ or $\hbar/2$, and its $z$-component will not have a definite value. There is no problem in understanding the change in the spin state of particle 1; measuring one spin component of this particle naturally affects other spin components. But if particle 1 and particle 2 are very far apart, then how can a measurement of the spin state of particle 1 affect the spin state of particle 2? And if it does not, then are we to conclude that the spin of particle 2 has definite values for both its $z$ and its $x$-components, even though these components do not commute? The only way to preserve consistency with quantum mechanics is to suppose that while the first measurement puts the system in a state where the first and second particles spin $z$-components are definite, the second measurement puts the system in a state where it is only the $x$-component of the first and second particles spin that have definite values. Though the particles are far apart, their spins remain entangled.

If I understand this description correctly, the author makes a serious mistake at some point, giving the reader an incorrect picture of entanglement. Can you find the mistake? Can you help rewrite the text so that it becomes correct, while still describing the EPR-paradox?