Solutions to problem set 6

6.1 Entanglement

a)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right)$$

This stateis a pure state, and thus has the density matrix:

$$\begin{aligned}\hat{\rho} &= \frac{1}{2}\left(|++\rangle\langle++| + |--\rangle\langle--| + |++\rangle\langle--| + |--\rangle\langle++|\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}|nn\rangle\langle mm|\end{aligned}$$

The entropy is then given by:

$$S_{\mathcal{A}} = S_{\mathcal{B}} = \text{Tr}_{\mathcal{A}}\left(\hat{\rho}_{\mathcal{A}}\log\hat{\rho}_{\mathcal{A}}\right)\ \left(= \text{Tr}_{\mathcal{B}}\left(\hat{\rho}_{\mathcal{B}}\log\hat{\rho}_{\mathcal{B}}\right)\right)$$

where $\hat{\rho}_{\mathcal{A}} = \text{Tr}_{\mathcal{B}}\left(\hat{\rho}\right)$. The trace of a matrix in the product space is:

$$\begin{aligned}\hat{\rho}_{\mathcal{A}} &= \text{Tr}_{\mathcal{B}}\left(\frac{1}{2}\sum_{n,m\in\{+,-\}}|nn\rangle\langle mm|\right) = \frac{1}{2}\sum_{n,m\in\{+,-\}}\text{Tr}_{\mathcal{B}}\left(|nn\rangle\langle mm|\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\text{Tr}_{\mathcal{B}}\left(\left(|n\rangle_{\mathcal{A}}\otimes|n\rangle_{\mathcal{B}}\right)\left(\langle m|_{\mathcal{A}}\otimes\langle m|_{\mathcal{B}}\right)\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\text{Tr}_{\mathcal{B}}\left(\left(|n\rangle\langle m|\right)_{\mathcal{A}}\otimes\left(|n\rangle\langle m|\right)_{\mathcal{B}}\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\left(|n\rangle\langle m|\right)_{\mathcal{A}}\otimes\text{Tr}\left(|n\rangle\langle m|\right)_{\mathcal{B}}\right)\\ &= \frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\left(|n\rangle\langle m|\right)_{\mathcal{A}}\otimes\delta_{mn}\right)\end{aligned}$$

Due to the trace only sums the diagonal elements ($\text{Tr}\left(|n\rangle\langle m|\right) = \langle m|n\rangle = \delta_{mn}$. Since δ_{mn} is a number, the tensor product reduces down to simple multiplication:

$$\hat{\rho}_{\mathcal{A}} = \frac{1}{2}\sum_{n,m\in\{+,-\}}\left(\delta_{mn}|n\rangle\langle m|\right)$$

Thus,

$$\hat{\rho}_{\mathcal{A}} = \frac{1}{2}\left(|+\rangle\langle+| + |-\rangle\langle-|\right)$$

This is a matrix with both eigenvalues $\frac{1}{2}$, thus we find the entropy:

$$S_{\mathcal{A}} = S_{\mathcal{B}} = -\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\ln\frac{1}{2} = \log 2 \tag{1}$$

Thus, they are maximally entangeled.

b) The operation $\hat{U}_B = \mathbb{1} \otimes \hat{U}_B$, and $\hat{U}_A = \hat{U}_A \otimes \mathbb{1}$, thus, applying both yields:

$$\hat{U}_A \hat{U}_B = \left(\hat{U}_A \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes \hat{U}_B\right) = \hat{U}_A \otimes \hat{U}_B$$

Applying this as a transformation, we get:

$$|\psi\rangle \to |\psi'\rangle = \hat{U}_A \otimes \hat{U}_B |\psi\rangle$$

$$\hat{\rho} \to \hat{\rho}' = \left(\hat{U}_A \otimes \hat{U}_B\right) \hat{\rho} \left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger$$

Then:

$$\begin{aligned}
\hat{\rho}'_A &= \mathrm{Tr}_B \left[\left(\hat{U}_A \otimes \hat{U}_B\right) \hat{\rho} \left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger\right] \\
&= \mathrm{Tr}_B \left[\left(\hat{U}_A \otimes \hat{U}_B\right) \left(\frac{1}{2} \sum_{n,m\in\{+,-\}} \left(|n\rangle_A \otimes |n\rangle_B\right)\left(\langle m|_A \otimes \langle m|_B\right)\right) \left(\hat{U}_A \otimes \hat{U}_B\right)^\dagger\right] \\
&= \mathrm{Tr}_B \left[\frac{1}{2} \sum_{n,m\in\{+,-\}} \left(\left[\hat{U}_A|n\rangle_A\right] \otimes \left[\hat{U}_B|n\rangle_B\right]\right)\left(\left[\langle m|_A \hat{U}_A^\dagger\right] \otimes \left[\langle m|_B \hat{U}_B^\dagger\right]\right)\right] \\
&= \frac{1}{2}\mathrm{Tr}_B \left[\sum_{n,m\in\{+,-\}} \left(\left[\hat{U}_A|n\rangle_A \langle m|_A \hat{U}_A^\dagger\right] \otimes \left[\hat{U}_B|n\rangle_B \langle m|_B \hat{U}_B^\dagger\right]\right)\right] \\
&= \frac{1}{2} \left[\sum_{n,m\in\{+,-\}} \left(\left[\hat{U}_A|n\rangle_A \langle m|_A \hat{U}_A^\dagger\right] \otimes \mathrm{Tr}\left[\hat{U}_B|n\rangle_B \langle m|_B \hat{U}_B^\dagger\right]\right)\right]
\end{aligned}$$

From problem set 1, we showed that $\mathrm{Tr}\left(\hat{U}A\hat{U}^\dagger\right) = \mathrm{Tr}\left(A\right)$ by

$$\mathrm{Tr}\left(\hat{U}A\hat{U}^\dagger\right) = \mathrm{Tr}\left(\hat{U}\left[A\hat{U}^\dagger\right]\right) = \mathrm{Tr}\left(\left[A\hat{U}^\dagger\right]\hat{U}\right) = \mathrm{Tr}\left(A\right)$$

We arrive at

$$\hat{\rho}'_A = \frac{1}{2}\left[\sum_{n,m\in\{+,-\}} \hat{U}_A|n\rangle\langle m|\hat{U}_A^\dagger \delta_{mn}\right] = \left[\sum_{n\in\{+,-\}} \hat{U}_A|n\rangle\langle n|\hat{U}_A^\dagger\right] = \hat{U}_A\hat{\rho}_A\hat{U}_A^\dagger$$

The entropy is then given as:

$$S'_A = -\mathrm{Tr}\left(\hat{\rho}'_A \log\left(\hat{\rho}'_A\right)\right)$$

Since $\hat{\rho}'_A = \hat{U}_A\hat{\rho}_A\hat{U}_A^\dagger$, they have the same eigenvalues, and therefore the entropy is the same.

c) After the measurement, part A of the system is projected on one of the eigenstates of the operator being measured (it does not matter which operator this is). It is then in a well defined pure state and not entangled with part B any more. The entropy of entanglement after the measurement is 0.

6.2 Schmidt decomposition 1

We have a system consisting of two spin-$\frac{1}{2}$ particles. For each of the following states, study the reduced density matrix of of one of the particles and determine if the state is entangled or not. For the states which are not entangled, find a factorization of the state as a tensor product of one state for each particle. For the entagled states, find the Schmidt decomposition of the state.

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
|\psi_2\rangle &= \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\
|\psi_3\rangle &= a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle \\
|\psi_4\rangle &= a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle
\end{aligned}
$$

where

$$a_\pm = \frac{\sqrt{3}\pm 1}{4}$$

$|\psi_1\rangle = \frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$**:**

The density matrix

$$\rho_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{4}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\uparrow| - \langle\uparrow\downarrow| + \langle\downarrow\uparrow| - \langle\downarrow\downarrow|\right)$$

$$\rho_1^A = \mathrm{Tr}_B\,\rho_1 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues are 0 and 1, which shows that $|\psi_1\rangle$ is not entangled. To find the factorization of the state we need the eigenvectors of the reduced density matrix ρ_1^A. The one with eigenvalue 1 is $|1\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$, while the one with eigenvalue 0 is $|0\rangle_A = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ (since this has eigenvalue 0 it will not appear in the factorization). We can now express the state $|\psi_1\rangle$ in terms of these eigenvectors and find that

$$|\psi_1\rangle = |1\rangle_A \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$|\psi_2\rangle = \frac{1}{2}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$**:**

The density matrix

$$\rho_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{4}\left(|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\uparrow| + \langle\uparrow\downarrow| + \langle\downarrow\uparrow| - \langle\downarrow\downarrow|\right)$$

$$\rho_2^A = \mathrm{Tr}_B\,\rho_2 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is not a pure state, so $|\psi_2\rangle$ is entangled. The eigenvalues are both $\frac{1}{2}$ and all vectors are eigenvectors. Because of that we can choose which basis to use for part A, and the Schmidt decomposition is not unique. Let us take the basis to be $|\uparrow\rangle$ and $|\downarrow\rangle$ for simplicity, and we find

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) + \frac{1}{\sqrt{2}}|\downarrow\rangle \otimes \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

$|\psi_3\rangle = a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle$**:**

The density matrix

$$\rho_3 = |\psi_3\rangle\langle\psi_3| = (a_+|\uparrow\uparrow\rangle + a_-|\uparrow\downarrow\rangle + a_-|\downarrow\uparrow\rangle + a_+|\downarrow\downarrow\rangle)\,(a_+\langle\uparrow\uparrow| + a_-\langle\uparrow\downarrow| + a_-\langle\downarrow\uparrow| + a_+\langle\downarrow\downarrow|)$$

$$\rho_3^A = \mathrm{Tr}_B\,\rho_3 = (a_+^2+a_-^2)|\uparrow\rangle\langle\uparrow| + 2a_+a_-|\uparrow\rangle\langle\downarrow| + 2a_+a_-|\downarrow\rangle\langle\uparrow| + (a_+^2+a_-^2)|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Diagonalizing we find the eigenvalues $p^+ = \frac{3}{4}$ with eigenvector $|\uparrow_x\rangle$ and $p^- = \frac{1}{4}$ with eigenvector $|\downarrow_x\rangle$. This is not a pure state, so $|\psi_3\rangle$ is entangled. Expressing the state in terms of the eigenvectors we find

$$|\psi_3\rangle = \frac{\sqrt{3}}{2}|\uparrow_x\uparrow_x\rangle + \frac{1}{2}|\downarrow_x\downarrow_x\rangle$$

$|\psi_4\rangle = a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle$**:**

The density matrix

$$\rho_4 = |\psi_3\rangle\langle\psi_3| = (a_-|\uparrow\uparrow\rangle + a_+|\uparrow\downarrow\rangle + a_+|\downarrow\uparrow\rangle + a_-|\downarrow\downarrow\rangle)\,(a_-\langle\uparrow\uparrow| + a_+\langle\uparrow\downarrow| + a_+\langle\downarrow\uparrow| + a_-\langle\downarrow\downarrow|)$$

$$\rho_4^A = \mathrm{Tr}_B\,\rho_4 = (a_+^2+a_-^2)|\uparrow\rangle\langle\uparrow| + 2a_+a_-|\uparrow\rangle\langle\downarrow| + 2a_+a_-|\downarrow\rangle\langle\uparrow| + (a_+^2+a_-^2)|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

which is the same as we found for ρ_3^A. Thus we get the same eigenvalues and eigenvectors and we find

$$|\psi_4\rangle = \frac{\sqrt{3}}{2}|\uparrow_x\uparrow_x\rangle - \frac{1}{2}|\downarrow_x\downarrow_x\rangle.$$

6.3 Schmidt decomposition 2

a) The Schmidt decomposition rewrites a general state in the product space, as a sum of states expressed in an orthonormal basis for each Hilbert space:

$$\Psi(x) = c_1\chi_1\phi_1(x) + c_2\chi_2\phi_2(x) \tag{2}$$

Thus, the spinors and wavefunctions must satisfy the orthonormality conditions

$$\chi_i^\dagger\chi_j = \int dx\phi_i^*\phi_j = \delta_{ij}$$

b) Normalization factor is given by $\langle\Psi|\Psi\rangle = 1$.

$$\begin{aligned} \langle\Psi|\Psi\rangle &= \int_{-\infty}^{\infty} |\,\Psi(x)\,|^2\,dx \\ &= \int_{-\infty}^{\infty} dx\,|\,\psi_1(x)\,|^2 + \int_{-\infty}^{\infty} dx\,|\,\psi_2(x)\,|^2 \\ &= |\,N\,|^2\left(\int_{-\infty}^{\infty} e^{-2\lambda(x-x_0)^2}dx + \int_{-\infty}^{\infty} e^{-2\lambda(x+x_0)^2}dx\right) \end{aligned}$$

Substituting $y = x \pm x_0$ in the first and second integral respectively yields :

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= 2 \mid N \mid^2 \int_{-\infty}^{\infty} e^{-2\lambda y^2} dx \\
&= 2 \mid N \mid^2 \sqrt{\frac{\pi}{2\lambda}} \\
\Rightarrow N &= \sqrt[4]{\frac{\lambda}{2\pi}}, \quad \text{when choosing } N \in \mathbb{R}
\end{aligned}$$

Then it follows:

$$\begin{aligned}
\Delta &= \langle\psi_1|\psi_2\rangle \\
&= N^2 \int_{-\infty}^{\infty} e^{-\lambda(x-x_0)^2} e^{-\lambda(x+x_0)^2} dx \\
&= N^2 \int_{-\infty}^{\infty} e^{-2\lambda\left(x^2+x_0^2\right)} dx \\
&= N^2 e^{-2\lambda x_0^2} \int_{-\infty}^{\infty} dx e^{-2\lambda x^2} \\
&= N^2 e^{-2\lambda x_0^2} \sqrt{\frac{\pi}{2\lambda}} \\
\Delta &= \frac{1}{2} e^{-2\lambda x_0^2}
\end{aligned}$$

c) To find the Schmidt decompostion, we have to find the eigenstates of the reduced density matrix for at least one of the subsystems (spin or position). It is simplest to work with spin, since it has the smallest Hilbert space. Therefore we will trace over the position

$$\rho_{spin} = \int dx \langle x|\Phi\rangle\langle\Phi|x\rangle = \int dx \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \begin{pmatrix} \psi_1^* & \psi_2^* \end{pmatrix} = \int dx \begin{pmatrix} \psi_1\psi_1^* & \psi_1\psi_2^* \\ \psi_2\psi_1^* & \psi_2\psi_2^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \Delta \\ \Delta & \frac{1}{2} \end{pmatrix}$$

The eigenvalues of this are

$$p_1 = \frac{1}{2}\left(1 + e^{-2\lambda x_0^2}\right) \qquad p_2 = \frac{1}{2}\left(1 - e^{-2\lambda x_0^2}\right)$$

with the corresponding eigenvectors

$$\chi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \chi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The coefficients in the Schmidt decomposition are the square roots of the eigenvectors, $c_i = \sqrt{p_i}$ and we get from Eq (2) that

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{2}} c_1 \phi_1 + \frac{1}{\sqrt{2}} c_2 \phi_2 \\
\psi_2 &= \frac{1}{\sqrt{2}} c_1 \phi_1 - \frac{1}{\sqrt{2}} c_2 \phi_2
\end{aligned}$$

which we can solve to find

$$\phi_1 = \frac{1}{\sqrt{2}c_1}(\psi_1 + \psi_2) = \frac{N}{\sqrt{1+e^{-2\lambda x_0^2}}}\left(e^{-\lambda(x-x_0)^2} + e^{-\lambda(x+x_0)^2}\right)$$
$$\phi_2 = \frac{1}{\sqrt{2}c_2}(\psi_1 - \psi_2) = \frac{N}{\sqrt{1-e^{-2\lambda x_0^2}}}\left(e^{-\lambda(x-x_0)^2} - e^{-\lambda(x+x_0)^2}\right)$$

6.4 Coupled two-level systems

$$\hat{H} = \frac{\epsilon}{2}\left(3\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_z\right) + \lambda\left(\sigma_+ \otimes \sigma_- + \sigma_- \otimes \sigma_+\right)$$
$$\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right)$$
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a)

$$\begin{aligned} \sigma_+ &= \frac{1}{2}\begin{pmatrix} 0 & 1+1 \\ 1-1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \sigma_- &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \hat{H} &= \frac{\epsilon}{2}\left[\begin{pmatrix} 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & -3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}\right] \qquad (3) \\ &\quad +\lambda\left[\begin{pmatrix} 0 & 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 2\epsilon & 0 & 0 & 0 \\ 0 & \epsilon & \lambda & 0 \\ 0 & \lambda & -\epsilon & 0 \\ 0 & 0 & 0 & -2\epsilon \end{pmatrix} \qquad (4) \end{aligned}$$

The eigenvalue equation becomes:

$$
\begin{aligned}
|\,\hat{H} - \mathbb{1}e\,| &= 0 \\
\begin{vmatrix} 2\epsilon - e & 0 & 0 & 0 \\ 0 & \epsilon - e & \lambda & 0 \\ 0 & \lambda & -\epsilon - e & 0 \\ 0 & 0 & 0 & -2\epsilon - e \end{vmatrix} &= 0 \\
(2\epsilon - e)\begin{vmatrix} \epsilon - e & \lambda & 0 \\ \lambda & -\epsilon - e & 0 \\ 0 & 0 & -2\epsilon - e \end{vmatrix} &= 0 \\
(2\epsilon - e)\,(-2\epsilon - e)\left[(\epsilon - e)\,(-\epsilon - e) - \lambda^2\right] &= 0
\end{aligned}
$$

From here, we immidiately see the value of the first two eigenvalues, the rest is determined by:

$$
\begin{aligned}
-\left(\epsilon^2 - e^2\right) - \lambda^2 &= 0 \\
e &= \pm\sqrt{\epsilon^2 + \lambda^2}
\end{aligned}
$$

The eigenvalues are thus:

$$
e_1 = 2\epsilon, \quad e_2 = -2\epsilon, \quad e_3 = \sqrt{\epsilon^2 + \lambda^2}, \quad e_4 = -\sqrt{\epsilon^2 + \lambda^2}
$$

We see that e_1 and e_2 are independent of λ, and from the hamiltonian (4), it is easy to to see that the eigenvectors are:

$$
\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

Then setting $\epsilon = \mu\cos\theta$ and $\lambda = \mu\sin\theta$, we get:

$$
e_1 = 2\mu\cos\theta, \quad e_2 = -2\mu\cos\theta, \quad e_3 = \mu, \quad e_4 = -\mu
$$

The hamiltonian takes the form:

$$
\hat{H} = \begin{pmatrix} 2\mu\cos\theta & 0 & 0 & 0 \\ 0 & \mu\cos\theta & \mu\sin\theta & 0 \\ 0 & \mu\sin\theta & -\mu\cos\theta & 0 \\ 0 & 0 & 0 & -2\mu\cos\theta \end{pmatrix}
$$

For the remaining subspace, the eigenvector equation is:

$$
\begin{pmatrix} \mu\cos\theta & \mu\sin\theta \\ \mu\sin\theta & -\mu\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\mu\begin{pmatrix} a \\ b \end{pmatrix}
$$

Then:

$$
\begin{aligned}
a\cos\theta + b\sin\theta &= \pm a \\
a\cos\theta - b\sin\theta &= \pm b
\end{aligned}
$$

Staring with the first equation:

$$a\cos\theta + b\sin\theta = \pm a \Rightarrow b = a\frac{\pm 1 - \cos\theta}{\sin\theta}$$

Then, if $a = \sin\theta$, we get the following eigenvectors:

$$\mathbf{e}_3' = \begin{pmatrix} 0 \\ \sin\theta \\ 1-\cos\theta \\ 0 \end{pmatrix}, \quad \mathbf{e}_4' = \begin{pmatrix} 0 \\ \sin\theta \\ -1-\cos\theta \\ 0 \end{pmatrix}$$

I marked them as to say that they are not the final eigenvectors, they need to be normalized first:

$$\begin{aligned} \sqrt{\mathbf{e}_3'\cdot\mathbf{e}_3'} &= \sqrt{\sin^2\theta + (1-\cos\theta)^2} = \sqrt{\sin^2\theta + 1 - 2\cos\theta + \cos\theta^2} = \sqrt{2-2\cos\theta} \\ \sqrt{\mathbf{e}_4'\cdot\mathbf{e}_4'} &= \sqrt{\sin^2\theta + (1+\cos\theta)^2} = \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta} = \sqrt{2+2\cos\theta} \end{aligned}$$

Then:

$$\mathbf{e}_3 = \frac{1}{\sqrt{2-2\cos\theta}}\begin{pmatrix} 0 \\ \sin\theta \\ 1-\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \end{pmatrix}, \quad \mathbf{e}_4 = \frac{1}{\sqrt{2+2\cos\theta}}\begin{pmatrix} 0 \\ \sin\theta \\ -1-\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \\ 0 \end{pmatrix}$$

Just to summarize, the eigenvectors are:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \\ 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \\ 0 \end{pmatrix}$$

The energies are:

$$E_1 = 2\mu\cos\theta, \quad E_2 = -2\mu\cos\theta, \quad E_3 = \mu, \quad E_4 = -\mu$$

b) The two interesting eigenstates are $\mathbf{e}_3$ and $\mathbf{e}_4$

$$\begin{aligned} \hat{\rho}_3 &= \mathbf{e}_3\mathbf{e}_3^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\frac{\theta}{2} & \cos\frac{\theta}{2}\sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \hat{\rho}_4 &= \mathbf{e}_4\mathbf{e}_4^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2\frac{\theta}{2} & -\cos\frac{\theta}{2}\sin\frac{\theta}{2} & 0 \\ 0 & -\cos\frac{\theta}{2}\sin\frac{\theta}{2} & \cos^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Before considering the partial traces, let's look at how this works out in the matrix representation. A general 4x4 matrix can be written as a sum over tensor products between 2x2 matrices (also called “Kronecker product”):

$$\begin{aligned} C &= \sum_{ij} c_{ij} A_i \otimes B_j \\ &= \sum_{ij} c_{ij} \begin{pmatrix} A^i_{11} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}_j & A^i_{12} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}_j \\ A^i_{21} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}_j & A^i_{22} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}_j \end{pmatrix} \\ &\equiv \sum_{ij} c_{ij} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}_{ij} \end{aligned}$$

Then the partial traces become:

$$\begin{aligned} \mathrm{Tr}_A C &= \sum_{ij} c_{ij} \left(C_{11} + C_{22} \right)_{ij} \\ \mathrm{Tr}_B C &= \sum_{ij} c_{ij} \begin{pmatrix} \mathrm{Tr} C_{11} & \mathrm{Tr} C_{12} \\ \mathrm{Tr} C_{21} & \mathrm{Tr} C_{22} \end{pmatrix}_{ij} \end{aligned}$$

And since $\mathrm{Tr}\,(A+B) = \mathrm{Tr}A + \mathrm{Tr}B$, we see that in our case:

$$\begin{aligned} \hat\rho_3^A = \mathrm{Tr}_B \hat\rho_3 &= \begin{pmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{pmatrix} \\ \hat\rho_3^B = \mathrm{Tr}_A \hat\rho_3 &= \begin{pmatrix} \sin^2\frac{\theta}{2} & 0 \\ 0 & \cos^2\frac{\theta}{2} \end{pmatrix} \\ \hat\rho_4^A = \mathrm{Tr}_B \hat\rho_4 &= \begin{pmatrix} \sin^2\frac{\theta}{2} & 0 \\ 0 & \cos^2\frac{\theta}{2} \end{pmatrix} \\ \hat\rho_4^B = \mathrm{Tr}_A \hat\rho_4 &= \begin{pmatrix} \cos^2\frac{\theta}{2} & 0 \\ 0 & \sin^2\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

c) We see that all the reduced density matrices have the same eigenvalues, and the von Neuman entropy is thus the same and given by:

$$S = -\cos^2\frac{\theta}{2} \ln \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \ln \sin^2\frac{\theta}{2}$$

The entropy is maximal when $\cos^2\frac{\theta}{2} = \sin^2\frac{\theta}{2} = \frac{1}{2}$, which means

$$\theta = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

6.5 Entanglement and measurements

The problem lies in the sentences ”Following this measurement, suppose that the x-component of the spin of particle 1 is measured. It will be found to have the value $\hbar/2$ or $-\hbar/2$, and the z-component of particle 1s spin will no longer have a definite value. Also, because the system has zero total angular momentum, the spin of particle 2 will then have x-component $-\hbar/2$ or $\hbar/2$, and its z-component will not have a definite value." It seems that it is assumed that first the spin is measured along z and then subsequently along x. But the first measurement along z will collapse the wavefunction, destroying all entanglement between the two particles. Measuring along x after that will give random uncorrelated results on the two particles, and not the perfect anticorrelation as stated. The appeal to "zero total angular momentum" is not relevant, as the interaction with the measuring device can change the angular momentum, as it does even when considering measuring the spin along different axes for a single particle. The text would be fine if instead of "Following this measurement, suppose..." we write "Supose instead...". This would mean that we can choose to measure either along z or x (but not both), and in both cases will we be able to deduce the corresponding spin component of the other particle. This component must then correspond to an element of reality according to EPR, and this is what they wanted to explain.