Fys 4110, 2019 Solutions

Problem 1

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)$$

a) $$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right)\left(\langle\uparrow\downarrow\downarrow| + \langle\downarrow\uparrow\downarrow| + \langle\downarrow\downarrow\uparrow|\right)$$

$$\rho_A = \mathrm{Tr}_{BC}\,\rho = \sum_{i,j=\uparrow,\downarrow} \langle i_B j_C|\rho|i_B j_C\rangle$$

$$= \frac{1}{3}\left(|\uparrow\rangle\langle\uparrow| + 2|\downarrow\rangle\langle\downarrow|\right)$$

$$\rho_B = \mathrm{Tr}_A\,\rho = \frac{1}{3}\left(|\downarrow\downarrow\rangle\langle\downarrow\downarrow| + |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\right)$$

$$S = -\mathrm{Tr}_A\,\rho_A \ln\rho_A = -\mathrm{Tr}_{BC}\,\rho_{BC}\ln\rho_{BC}$$ Easiest to use ρ_A

$$S = -\frac{1}{3}\ln\frac{1}{3} - \frac{2}{3}\ln\frac{2}{3}$$

b) Measure $\uparrow$: $|\psi\rangle \to |\uparrow\downarrow\downarrow\rangle$ $\qquad S_{BC} = 0$

Measure $\downarrow$: $|\psi\rangle \to \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ $\qquad S'_{BC} = \ln 2$

c) Eigenstates for σ_x:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \qquad \sigma_x|\rightarrow\rangle = |\rightarrow\rangle$$

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \qquad \sigma_x|\leftarrow\rangle = -|\leftarrow\rangle$$

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle + |\leftarrow\rangle\right) \qquad |\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle - |\leftarrow\rangle\right)$$

$$|\psi\rangle = \frac{1}{\sqrt{6}}\left(|\rightarrow\downarrow\downarrow\rangle + |\leftarrow\downarrow\downarrow\rangle + |\rightarrow\uparrow\downarrow\rangle - |\leftarrow\uparrow\downarrow\rangle + |\rightarrow\downarrow\uparrow\rangle - |\leftarrow\downarrow\uparrow\rangle\right)$$

Measure $\rightarrow$: $|\psi\rangle \to |\rightarrow\rangle\frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

Measure $\leftarrow$: $|\psi\rangle \to |\leftarrow\rangle\frac{1}{\sqrt{3}}\left(|\downarrow\downarrow\rangle - |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

For BC we have

$$|\psi_{BC}\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\right)$$

$$\rho_{BC} = |\psi_{BC}\rangle\langle\psi_{BC}| = \frac{1}{3}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \pm |\downarrow\downarrow\rangle\right)\left(\langle\uparrow\downarrow| + \langle\downarrow\uparrow| \pm \langle\downarrow\downarrow|\right)$$

$$\rho_B = \mathrm{Tr}_C\, \rho_{BC} = \frac{1}{3}\left(2|\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| \pm |\uparrow\rangle\langle\downarrow| \pm |\downarrow\rangle\langle\uparrow|\right)$$

In matrix form $|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$\rho_B = \frac{1}{3}\begin{pmatrix}1 & \pm 1\\ \pm 1 & 2\end{pmatrix}$$

Eigenvalues $\begin{vmatrix}\frac{1}{3}-\lambda & \pm\frac{1}{3}\\ \pm\frac{1}{3} & \frac{2}{3}-\lambda\end{vmatrix} = (\lambda-\frac{1}{3})(\lambda-\frac{2}{3}) - \frac{1}{9} = 0$

$$\Rightarrow (3\lambda-1)(3\lambda-2)-1 = 9\lambda^2 - 9\lambda - 1 = 0 \quad \Rightarrow \lambda_\pm = \frac{9\pm\sqrt{81+36}}{18} = \frac{1\pm\sqrt{13}}{2}$$

Entanglement entropy: $S = -\frac{1+\sqrt{13}}{2}\ln\frac{1+\sqrt{13}}{2} - \frac{1-\sqrt{13}}{2}\ln\frac{1-\sqrt{13}}{2}$

Problem 2.

$$H = \frac{\hbar}{2}\omega_0\sigma_z + \frac{\hbar}{2}A(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y)$$

a) $i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle \qquad |\psi'\rangle = e^{i\frac{\omega t}{2}\sigma_z}|\psi\rangle$

$$i\hbar\frac{d}{dt}|\psi'\rangle = i\hbar\left(i\frac{\omega}{2}\sigma_z|\psi'\rangle + e^{i\frac{\omega t}{2}\sigma_z}\frac{d}{dt}|\psi\rangle\right)$$

$$= \Big(\underbrace{-\frac{\hbar}{2}\omega\sigma_z + e^{i\frac{\omega t}{2}\sigma_z}He^{-i\frac{\omega t}{2}\sigma_z}}_{H'}\Big)|\psi'\rangle$$

$$e^{i\frac{\omega t}{2}\sigma_z}\sigma_x e^{-i\frac{\omega t}{2}\sigma_z} = \left(\cos\frac{\omega t}{2}\mathbb{1} + i\sin\frac{\omega t}{2}\sigma_z\right)\sigma_x\left(\cos\frac{\omega t}{2}\mathbb{1} - i\sin\frac{\omega t}{2}\sigma_z\right)$$

$$= \cos^2\frac{\omega t}{2}\sigma_x + i\cos\frac{\omega t}{2}\sin\frac{\omega t}{2}\underbrace{[\sigma_z,\sigma_x]}_{2i\sigma_y} + \sin^2\frac{\omega t}{2}\underbrace{\underbrace{\sigma_z\sigma_x}_{i\sigma_y}\sigma_z}_{-\sigma_x}$$

$$= \cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y$$

$$e^{i\frac{\omega t}{2}\sigma_z}\sigma_y e^{-i\frac{\omega t}{2}\sigma_z} = \left(\cos\frac{\omega t}{2}\mathbb{1} + i\sin\frac{\omega t}{2}\sigma_z\right)\sigma_y\left(\cos\frac{\omega t}{2}\mathbb{1} - i\sin\frac{\omega t}{2}\sigma_z\right)$$

$$= \cos\omega t\,\sigma_y + \sin\omega t\,\sigma_x$$

$$H' = -\frac{\hbar}{2}\omega\sigma_z + \frac{\hbar}{2}\omega_0\sigma_z + \frac{\hbar}{2}A\left[\cos^2\omega t\,\sigma_x - \cos\omega t\sin\omega t\,\sigma_y + \cos\omega t\sin\omega t\,\sigma_y + \sin^2\omega t\,\sigma_x\right]$$

$$= \frac{\hbar}{2}(\omega_0-\omega)\sigma_z + \frac{\hbar}{2}A\sigma_x \qquad \text{Time independent}$$

Resonance when $\omega = \omega_0$.

b) $$H' = \frac{\hbar}{2}(\omega_0-\omega)\sigma_z + \frac{\hbar}{2}A\left[\underbrace{\cos^2\omega t}_{\frac{1}{2}(1+\cos 2\omega t)}\sigma_x - \underbrace{\cos\omega t\sin\omega t}_{\frac{1}{2}\sin 2\omega t}\sigma_y\right]$$

$$= \frac{\hbar}{2}(\omega_0-\omega)\sigma_z + \frac{\hbar}{4}A\sigma_x + \frac{\hbar A}{4}\underbrace{\left(\cos 2\omega t\,\sigma_x - \sin 2\omega t\,\sigma_y\right)}_{\text{Rotating with frequency } 2\omega}$$

The oscillating field $\cos\omega t\,\sigma_x$ can be thought of as two counterrotating fields

$$\cos\omega t\,\sigma_x = \frac{1}{2}(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y) + \frac{1}{2}(\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y)$$

When transforming to the rotating frame, the first term will appear constant while the second term will appear as rotating at twice the frequency.

We can neglect the term $\frac{\hbar A}{4}(\cos 2\omega t\, \sigma_x - \sin 2\omega t\, \sigma_y)$ when A is sufficiently small because it changes rapidly in time and its effect on the state does not have time to build up before it changes direction. On average it does not have large effect, and the true state will wiggle around the approximate state that we find using the rotating wave approximation.

c) $H' = -\hbar \frac{dS}{dt} + e^{iS} H e^{-iS}$

$$S = \frac{A}{2\omega} \xi \sin\omega t\, \sigma_x = \hat{A}\, \sigma_x$$

$$\frac{dS}{dt} = \frac{A}{2} \xi \cos\omega t\, \sigma_x$$

$$e^{iS}\sigma_z e^{-iS} = e^{i\hat{A}\sigma_x}\sigma_z e^{-i\hat{A}\sigma_x} = (\cos\hat{A}\,\mathbb{1} + i\sin\hat{A}\,\sigma_x)\,\sigma_z\,(\cos\hat{A}\,\mathbb{1} - i\sin\hat{A}\,\sigma_x)$$

$$= \cos^2\hat{A}\,\sigma_z + i\cos\hat{A}\sin\hat{A}\,\underbrace{[\sigma_x,\sigma_z]}_{-2i\sigma_y} + \sin^2\hat{A}\,\underbrace{\sigma_x\sigma_z\sigma_x}_{-\sigma_z}$$

$$= \cos 2\hat{A}\,\sigma_z + \sin 2\hat{A}\,\sigma_y$$

$$H' = -\frac{\hbar A}{2}\xi\cos\omega t\,\sigma_x + \frac{\hbar}{2}\omega_0\cos\left[\frac{A}{\omega}\xi\sin\omega t\right]\sigma_z + \frac{\hbar}{2}\omega_0\sin\left[\frac{A}{\omega}\xi\sin\omega t\right]\sigma_y$$

$$+ \frac{\hbar}{2}A\cos\omega t\,\sigma_x$$

$$= \frac{\hbar}{2}\omega_0\left\{\cos\left[\frac{A}{\omega}\xi\sin\omega t\right]\sigma_z + \sin\left[\frac{A}{\omega}\xi\sin\omega t\right]\sigma_y\right\} + \frac{\hbar}{2}A(1-\xi)\cos\omega t\,\sigma_x$$

d) If $\quad J_1\left(\frac{A}{\omega}\xi\right)\omega_0 = \frac{1}{2}A(1-\xi) = \frac{1}{2}A'$ we have

$$H' \approx \frac{\hbar}{2}\omega_0 J_0\left(\frac{A}{\omega}\xi\right)\sigma_z + \frac{\hbar}{2}A'\left(\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y\right)$$

With this choice of ξ, the components of the field in the x- and y-directions have the same amplitude, and we have a rotating field similar to that in question a) but with ω_0 rescaled by the Bessel function. The resonance condition is therefore $\quad \omega = \omega_0 J_0\left(\frac{A}{\omega}\xi\right)$

e) $\quad J_1\left(\frac{A}{\omega}\xi\right)\omega_0 \approx \frac{A}{2\omega}\xi\,\omega_0 = \frac{1}{2}A(1-\xi)$

$$\Rightarrow \xi = \frac{1}{1+\frac{\omega_0}{\omega}} = \frac{\omega}{\omega_0+\omega}$$

$$\omega = \omega_0 J_0\left(\frac{A}{\omega}\xi\right) = \omega_0 J_0\left(\frac{A}{\omega_0+\omega}\right) \approx \omega_0\left(1 - \frac{A^2}{4(\omega_0+\omega)^2}\right)$$

For $A=0$ we have $\omega=\omega_0$ and in general $\omega = \omega_0 + (\,)A^2$

To lowest order we can then replace $\omega_0+\omega \rightarrow 2\omega_0$ in the denominator to get

$$\omega = \omega_0 - \frac{A^2}{16\,\omega_0}$$

Problem 3

a) $P(\theta,\phi) = N\sum_a |(\vec{k}\times\vec{\epsilon}_{ka})\cdot\vec{\sigma}_{BA}|^2$ where N is a normalization to be determined at the end.

$$\vec{\sigma}_{BA} = \langle\downarrow|\vec{\sigma}|\uparrow\rangle = \left((0\,1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, (0\,1)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, (0\,1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right)$$
$$= (1, i, 0)$$

We have $\vec{k}\times\vec{\epsilon}_{k1} = \underset{|\vec{k}|}{k}\,\vec{\epsilon}_{k2}$ $\quad \vec{k}\times\vec{\epsilon}_{k2} = -k\,\vec{\epsilon}_{k1}$

$$\Rightarrow \sum_a |(\vec{k}\times\vec{\epsilon}_{ka})\cdot\vec{\sigma}_{BA}|^2 = k^2\sum_a |\vec{\epsilon}_{ka}\cdot\vec{\sigma}_{BA}|^2 = k^2\left(|\vec{\sigma}_{BA}|^2 - |\vec{\sigma}_{BA}\cdot\tfrac{\vec{k}}{k}|^2\right)$$

$\frac{\vec{k}}{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$\vec{\sigma}_{BA}\cdot\frac{\vec{k}}{k} = \sin\theta\, e^{i\phi}$ $\qquad |\vec{\sigma}_{BA}|^2 = 2$

$$\Rightarrow p(\theta,\phi) = Nk^2(2-\sin^2\theta) = Nk^2(1+\cos^2\theta)$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, p(\theta,\phi) = Nk^2\cdot 2\pi \int_0^{\pi} d\theta\sin\theta(1+\cos^2\theta) \qquad u = \cos\theta$$

$$= 2\pi Nk^2 \underbrace{\int_{-1}^{1} du(1+u^2)}_{2+2/3 = \frac{8}{3}} = \frac{16\pi}{3}Nk^2 = 1 \quad\Rightarrow N = \frac{3}{16\pi k^2}$$

$$\Rightarrow p(\theta,\phi) = \frac{3}{16\pi}(1+\cos^2\theta)$$

b) $\vec{k} = (1,0,0)$ $\qquad \vec{\epsilon}_{k1} = (0, \cos\alpha, \sin\alpha)$

$$p(\alpha) = N|\underbrace{(\vec{k}\times\vec{\epsilon}_{k1})}_{(0,-\sin\alpha,\cos\alpha)}\cdot\vec{\sigma}_{BA}|^2 = N\sin^2\alpha$$

$$\int_0^{2\pi} p(\alpha)d\alpha = N\int_0^{2\pi}\sin^2\alpha\, d\alpha = N\pi = 1 \quad\Rightarrow\quad N = \frac{1}{\pi}$$

$$\Rightarrow p(\alpha) = \frac{1}{\pi}\sin^2\alpha$$

It is equally reasonable to restrict $0 \le \alpha \le \pi$, since α and $\alpha+\pi$ give the same polarization state, and normalize according to $\int_0^{\pi} d\alpha\, p(\alpha) = 1$

$$\Rightarrow p(\alpha) = \frac{2}{\pi}\sin^2\alpha$$

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e) $$w_{BA} = \frac{V}{(2\pi\hbar)^2} \int d^3k \sum_{a} |\langle B, 1_{ka} | H_I | A, 0\rangle|^2 \, \delta(\omega - \omega_B)$$

$$= \frac{V}{(2\pi\hbar)^2} \frac{e^2\hbar^2}{4m^2} \frac{\hbar}{2V\epsilon_0} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} k^2 dk \, \frac{1}{\omega} \, \delta(\omega - \omega_B) \cdot \underbrace{\sum_{a} |(\vec{k} \times \vec{\epsilon}_{ka}) \cdot \vec{\sigma}_{BA}|^2}_{P(\theta,\phi)/N \, = \, k^2(1+\cos^2\theta)}$$

$$= \frac{e^2\hbar}{32\pi^2 m^2 \epsilon_0 c^5} \, 2\pi \cdot \underbrace{\int_0^{\pi} d\theta \, (1+\cos^2\theta) \sin\theta}_{8/3} \int_0^{\infty} \omega^3 d\omega \, \delta(\omega - \omega_B)$$

$$= \frac{e^2 \hbar \, \omega_B^3}{6\pi m^2 \epsilon_0 c^5}$$

$$\tau = \frac{1}{w_{BA}} = \frac{6\pi m^2 \epsilon_0 c^5}{e^2 \hbar \, \omega_B^3}$$