

Solutions to problem set 6

6.1 The canonical commutation relations.

$$\hat{A}_{\mathbf{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}} (\hat{a}_{\mathbf{k}a} + \hat{a}_{-\mathbf{k}\bar{a}}^\dagger), \quad \hat{E}_{\mathbf{k}a} = i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}} (\hat{a}_{\mathbf{k}a} - \hat{a}_{-\mathbf{k}\bar{a}}^\dagger)$$

$$[\hat{A}_{\mathbf{k}a}, \hat{E}_{\mathbf{k}'a'}^\dagger] = -i\frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}\mathbf{k}'} \delta_{aa'}$$

We need to express the ladder operators in terms of \hat{A} and \hat{E}^\dagger , by inspection, we see that:

$$\hat{a}_{\mathbf{k}a} = \frac{1}{2} \left(\sqrt{\frac{2\omega_k\epsilon_0}{\hbar}} \hat{A}_{\mathbf{k}a} - i\sqrt{\frac{2\epsilon_0}{\hbar\omega_k}} \hat{E}_{\mathbf{k}a} \right)$$

from which we calculate:

$$\hat{a}_{\mathbf{k}a}^\dagger = \frac{1}{2} \left(\sqrt{\frac{2\omega_k\epsilon_0}{\hbar}} \hat{A}_{\mathbf{k}a}^\dagger + i\sqrt{\frac{2\epsilon_0}{\hbar\omega_k}} \hat{E}_{\mathbf{k}a}^\dagger \right)$$

Then:

$$\begin{aligned} [\hat{a}_{\mathbf{k}a}, \hat{a}_{\mathbf{k}'a'}^\dagger] &= \left[\frac{1}{2} \left(\sqrt{\frac{2\omega_k\epsilon_0}{\hbar}} \hat{A}_{\mathbf{k}a} - i\sqrt{\frac{2\epsilon_0}{\hbar\omega_k}} \hat{E}_{\mathbf{k}a} \right), \frac{1}{2} \left(\sqrt{\frac{2\omega_{k'}\epsilon_0}{\hbar}} \hat{A}_{\mathbf{k}'a'}^\dagger + i\sqrt{\frac{2\epsilon_0}{\hbar\omega_{k'}}} \hat{E}_{\mathbf{k}'a'}^\dagger \right) \right] \\ &= \frac{1}{4} \left(\frac{2\epsilon_0\sqrt{\omega_k\omega_{k'}}}{\hbar} \underbrace{[\hat{A}_{\mathbf{k}a}, \hat{A}_{\mathbf{k}'a'}^\dagger]}_{=0} + i\frac{2\epsilon_0}{\hbar} \sqrt{\frac{\omega_k}{\omega_{k'}}} [\hat{A}_{\mathbf{k}a}, \hat{E}_{\mathbf{k}'a'}^\dagger] \right) \\ &\quad + \frac{1}{4} \left(-i\frac{2\epsilon_0}{\hbar} \sqrt{\frac{\omega_{k'}}{\omega_k}} [\hat{E}_{\mathbf{k}a}, \hat{A}_{\mathbf{k}'a'}^\dagger] + \frac{2\epsilon_0}{\hbar\sqrt{\omega_k\omega_{k'}}} \underbrace{[\hat{E}_{\mathbf{k}a}, \hat{E}_{\mathbf{k}'a'}^\dagger]}_{=0} \right) \\ &= i\frac{\epsilon_0}{2\hbar} \sqrt{\frac{\omega_k}{\omega_{k'}}} \underbrace{[\hat{A}_{\mathbf{k}a}, \hat{E}_{\mathbf{k}'a'}^\dagger]}_{=-i\frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}\mathbf{k}'} \delta_{aa'}} - i\frac{\epsilon_0}{2\hbar} \sqrt{\frac{\omega_{k'}}{\omega_k}} [\hat{E}_{\mathbf{k}a}, \hat{A}_{\mathbf{k}'a'}^\dagger] \end{aligned}$$

Looking at the last commutator:

$$\begin{aligned} [\hat{E}_{\mathbf{k}a}, \hat{A}_{\mathbf{k}'a'}^\dagger] &= \hat{E}_{\mathbf{k}a} \hat{A}_{\mathbf{k}'a'}^\dagger - \hat{A}_{\mathbf{k}'a'}^\dagger \hat{E}_{\mathbf{k}a} = (\hat{A}_{\mathbf{k}'a'} \hat{E}_{\mathbf{k}a}^\dagger)^\dagger - (\hat{E}_{\mathbf{k}a}^\dagger \hat{A}_{\mathbf{k}'a'})^\dagger \\ &= (\hat{A}_{\mathbf{k}'a'} \hat{E}_{\mathbf{k}a}^\dagger - \hat{E}_{\mathbf{k}a}^\dagger \hat{A}_{\mathbf{k}'a'})^\dagger = [\hat{A}_{\mathbf{k}'a'}, \hat{E}_{\mathbf{k}a}^\dagger]^\dagger \\ &= \left(-i\frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}'\mathbf{k}} \delta_{a'a} \right)^\dagger = i\frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}'\mathbf{k}} \delta_{a'a} \end{aligned}$$

So far, we have:

$$\begin{aligned} [\hat{a}_{\mathbf{k}a}, \hat{a}_{\mathbf{k}'a'}^\dagger] &= i \frac{\epsilon_0}{2\hbar} \sqrt{\frac{\omega_k}{\omega_{k'}}} \left(-i \frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}\mathbf{k}'} \delta_{aa'} \right) - i \frac{\epsilon_0}{2\hbar} \sqrt{\frac{\omega_{k'}}{\omega_k}} \left(i \frac{\hbar}{\epsilon_0} \delta_{\mathbf{k}'\mathbf{k}} \delta_{a'a} \right) \\ &= \frac{1}{2} \sqrt{\frac{\omega_k}{\omega_{k'}}} \delta_{\mathbf{k}\mathbf{k}'} \delta_{aa'} + \frac{1}{2} \sqrt{\frac{\omega_{k'}}{\omega_k}} (\delta_{\mathbf{k}'\mathbf{k}} \delta_{a'a}) \\ &= \frac{1}{2} \left(\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}} \right) \delta_{\mathbf{k}\mathbf{k}'} \delta_{aa'} \end{aligned}$$

$\delta_{\mathbf{k}\mathbf{k}'}$ will either return zero if $\mathbf{k} \neq \mathbf{k}'$, or 1 if $\mathbf{k} = \mathbf{k}'$, in the latter case, $\sqrt{\frac{\omega_k}{\omega_{k'}}} + \sqrt{\frac{\omega_{k'}}{\omega_k}} = 2$, and if not, the expression has no contribution, thus, we can neglect it and get the desired result:

$$[\hat{a}_{\mathbf{k}a}, \hat{a}_{\mathbf{k}'a'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{aa'}$$

6.2 Charged particle in a strong magnetic field (Midterm Exam 2005).

a) From Newtons second law:

$$m \frac{d\mathbf{v}}{dt} = e(\mathbf{v} \times \mathbf{B}) \quad (1)$$

for a particle moving in a magnetic field ($\mathbf{E} = 0$). The velocity is restricted to the xy -plane, and the magnetic field is in the z -direction. Thus, by integration:

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{eB}{m} (\mathbf{v} \times \mathbf{k}) \\ \Rightarrow \mathbf{v} &= \frac{eB}{m} (\mathbf{r} \times \mathbf{k}) + \mathbf{C} \\ &= -\frac{eB}{m} \mathbf{k} \times \mathbf{r} + \mathbf{C} \end{aligned}$$

We recognize this as the expression for angular velocity with $\omega = -\frac{eB}{m}$ where \mathbf{C} is a constant that can be determined from the initial conditions. We can parametrize \mathbf{C} to a vector on the same form $\mathbf{C} = -\omega \times \mathbf{r}_0$ where \mathbf{r}_0 is a constant:

$$\begin{aligned} \mathbf{v} &= \vec{\omega} \times \mathbf{r} - \vec{\omega} \times \mathbf{r}_0 \\ \mathbf{v} &= \vec{\omega} \times (\mathbf{r} - \mathbf{r}_0) \end{aligned} \quad (2)$$

We see that this represents constant angular motion around the centre \mathbf{r}_0 with angular frequency $\omega = -\frac{eB}{m}$.

To check if $L_{mek} = m(xv_y - yv_x)$ is a constant of motion, we start by calculating:

$$\begin{aligned} m \frac{d\mathbf{v}}{dt} &= e\mathbf{v} \times \mathbf{B} \\ &= q \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & 0 \\ 0 & 0 & B \end{vmatrix} \\ &= v_y B \hat{i} - v_x B \hat{j} \end{aligned}$$

Then:

$$\begin{aligned}
 \frac{dL_{mek}}{dt} &= m \left(\frac{dx}{dt} v_y + x \frac{dv_y}{dt} - \frac{dy}{dt} v_x - y \frac{dv_x}{dt} \right) \\
 &= m \frac{dx}{dt} v_y + x m \frac{dv_y}{dt} - m \frac{dy}{dt} v_x - y m \frac{dv_x}{dt} \\
 &= m v_x v_y - e B x v_x - m v_y v_x - e B y v_y \\
 &= -e B (x v_x + y v_y) \\
 &= -e B \mathbf{r} \cdot \mathbf{v} \\
 &= -e B \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \right) \\
 &= -\frac{eB}{2} \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} \right) \\
 &= -\frac{eB}{2} \frac{d}{dt} r^2
 \end{aligned}$$

As we see, $\frac{d}{dt} L_{mek} \neq 0$, and thus not a constant of motion. Instead we have that $L = L_{mek} + (eB/2)r^2$ is conserved as:

$$\begin{aligned}
 \frac{dL}{dt} &= \frac{dL_{mek}}{dt} + \frac{eB}{2} \frac{d}{dt} r^2 \\
 &= \frac{eB}{2} \frac{d}{dt} (-r^2 + r^2) \\
 &= 0
 \end{aligned}$$

b) To check if \mathbf{R} is a constant of motion, we take the derivative:

$$\begin{aligned}
 \frac{d\mathbf{R}}{dt} &= \frac{d\mathbf{r}}{dt} + \frac{1}{\omega} \frac{d}{dt} (\mathbf{k} \times \mathbf{v}) \\
 &= \frac{d\mathbf{r}}{dt} + \frac{1}{\omega} \mathbf{k} \times \frac{d\mathbf{v}}{dt} \\
 &\stackrel{(1)}{=} \mathbf{v} + \frac{e}{m\omega} \mathbf{k} \times (\mathbf{v} \times \mathbf{B}) \\
 &= \mathbf{v} + \frac{e}{m\omega} (\mathbf{v} (\mathbf{k} \cdot \mathbf{B}) - \mathbf{B} (\mathbf{k} \cdot \mathbf{v})) \\
 &= \mathbf{v} + \frac{eB}{m\omega} \mathbf{v} \\
 &= \mathbf{v} + \frac{eB}{m} \frac{m}{-eB} \mathbf{v} \\
 &= \mathbf{0}
 \end{aligned}$$

Which it is. Inserting (2) into the expression for \mathbf{R} we have:

$$\begin{aligned}
 \mathbf{R} &= \mathbf{r} + \frac{1}{\omega} \mathbf{k} \times \mathbf{v} \\
 &= \mathbf{r} + \frac{1}{\omega} \mathbf{k} \times \vec{\omega} \times (\mathbf{r} - \mathbf{r}_0) \\
 &= \mathbf{r} + \frac{1}{\omega} \left(\vec{\omega} \cdot \underbrace{(\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_0))}_{=0} - (\mathbf{r} - \mathbf{r}_0) \underbrace{(\mathbf{k} \cdot \vec{\omega})}_{=\omega} \right) \\
 &= \mathbf{r} - (\mathbf{r} - \mathbf{r}_0) \\
 \mathbf{R} &= \mathbf{r}_0
 \end{aligned} \tag{3}$$

So \mathbf{R} points to the centre of the circular orbit. $\vec{\rho}$ is given by:

$$\begin{aligned}
 \vec{\rho} &= \mathbf{R} - \mathbf{r} \\
 &= \mathbf{r}_0 - \mathbf{r}
 \end{aligned}$$

So ρ points from the particle to the centre of orbit.

c) If we use (from the problem set):

$$\begin{aligned}
 \mathbf{v} &= \frac{(\mathbf{p} - e\mathbf{A})}{m} \\
 &= \frac{1}{m} \left(\mathbf{p} + \frac{e}{2} \mathbf{r} \times \mathbf{B} \right) \\
 &= \frac{1}{m} \left(\mathbf{p} + \frac{eB}{2} \mathbf{r} \times \mathbf{k} \right)
 \end{aligned}$$

where \mathbf{p} denotes the canonical momentum, we can express \mathbf{R} with \mathbf{p} and \mathbf{r} only :

$$\begin{aligned}
 \mathbf{R} &= \mathbf{r} + \frac{1}{\omega} \mathbf{k} \times \mathbf{v} \\
 &= \mathbf{r} + \frac{1}{m\omega} \mathbf{k} \times \left(\mathbf{p} + \frac{eB}{2} \mathbf{r} \times \mathbf{k} \right) \\
 &= \mathbf{r} + \frac{1}{m\omega} \left(\mathbf{k} \times \mathbf{p} + \frac{eB}{2} \underbrace{\mathbf{k} \times [\mathbf{r} \times \mathbf{k}]}_{=\mathbf{r}} \right) \\
 &= \mathbf{r} + \frac{1}{m\omega} \underbrace{\mathbf{k} \times \mathbf{p}}_{=-p_y \hat{i} + p_x \hat{j}} + \frac{eB}{2m\omega} \mathbf{r} \\
 &= \mathbf{r} \left(1 - \frac{1}{2} \right) + \frac{1}{m\omega} (-p_y \hat{i} + p_x \hat{j}) \\
 &= \left(\frac{1}{2}x - \frac{1}{m\omega} p_y \right) \mathbf{i} + \left(\frac{1}{2}y + \frac{1}{m\omega} p_x \right) \mathbf{j} \\
 &\equiv X\mathbf{i} + Y\mathbf{j}
 \end{aligned}$$

We can now express these as QM operators by replacing $r \rightarrow \hat{r}$ and $p \rightarrow \hat{p}$ with the commutation relations

$$[\hat{r}_j, \hat{p}_k] = i\hbar \delta_{jk}$$

This gives:

$$\begin{aligned}\hat{X} &= \frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y \\ \hat{Y} &= \frac{1}{2}\hat{y} - \frac{1}{m\omega}\hat{p}_x\end{aligned}$$

These commute as:

$$\begin{aligned}[\hat{X}, \hat{Y}] &= \left[\frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y, \frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x \right] \\ &= \frac{1}{4}\underbrace{[\hat{x}, \hat{y}]}_{=0} + \frac{1}{2m\omega}\underbrace{[\hat{x}, \hat{p}_x]}_{=i\hbar} - \frac{1}{2m\omega}\underbrace{[\hat{p}_y, \hat{y}]}_{=-i\hbar} - \frac{1}{m^2\omega^2}\underbrace{[\hat{p}_y, \hat{p}_x]}_{=0} \\ &= \frac{i\hbar}{m\omega}\end{aligned}$$

For $\rho = \mathbf{R} - \mathbf{r}$ we have the component operators:

$$\begin{aligned}\hat{\rho}_x &= -\frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y \\ \hat{\rho}_y &= -\frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x\end{aligned}$$

That gives:

$$\begin{aligned}[\rho_x, \rho_y] &= \left[-\frac{1}{2}\hat{x} - \frac{1}{m\omega}\hat{p}_y, -\frac{1}{2}\hat{y} + \frac{1}{m\omega}\hat{p}_x \right] \\ &= \frac{1}{4}\underbrace{[\hat{x}, \hat{y}]}_{=0} - \frac{1}{2m\omega}\underbrace{[\hat{x}, \hat{p}_x]}_{=i\hbar} + \frac{1}{2m\omega}\underbrace{[\hat{p}_y, \hat{y}]}_{=-i\hbar} - \frac{1}{m^2\omega^2}\underbrace{[\hat{p}_y, \hat{p}_x]}_{=0} \\ &= -\frac{i\hbar}{m\omega}\end{aligned}$$

Here \hat{X} and \hat{Y} and $\hat{\rho}_x$ and $\hat{\rho}_y$ respectively commute as a phase space where we have replaced $\hbar \rightarrow \hbar/m\omega$. This means that there are uncertainty relations between the operators, and that they can not be known simultaneously. We now introduce $l_B^2 = \hbar/m\omega$ such that

$$[\hat{X}, \hat{Y}] = [\hat{\rho}_y, \hat{\rho}_x] = il_B^2 \tag{4}$$

d)

$$\hat{a} = \frac{1}{\sqrt{2}l_B} (\hat{X} + i\hat{Y}) \quad \hat{b} = \frac{1}{\sqrt{2}l_B} (\hat{\rho}_x - i\hat{\rho}_y)$$

We know that \hat{X}, \hat{Y} and $\hat{\rho}_x, \hat{\rho}_y$ are made up of hermitian operators, and thus:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}l_B} (\hat{X} - i\hat{Y}), \quad \hat{b}^\dagger = \frac{1}{\sqrt{2}l_B} (\hat{\rho}_x + i\hat{\rho}_y)$$

where $l_B = \sqrt{\frac{\hbar}{|eB|}}$. Then:

$$\begin{aligned}
 [\hat{a}, \hat{a}^\dagger] &= \frac{1}{2l_B^2} [\hat{X} + i\hat{Y}, \hat{X} - i\hat{Y}] \\
 &= \frac{1}{2l_B^2} \left(\underbrace{[\hat{X}, \hat{X}]}_{=0} + \underbrace{[\hat{X}, -i\hat{Y}] + [i\hat{Y}, \hat{X}]}_{=-2i[\hat{X}, \hat{Y}]} + \underbrace{[i\hat{Y}, -i\hat{Y}]}_{=0} \right) \\
 &= \frac{-2i^2 l_B^2}{2l_B^2} \\
 &= 1 \\
 [\hat{b}, \hat{b}^\dagger] &= \frac{1}{2l_B^2} [\hat{\rho}_x - i\hat{\rho}_y, \hat{\rho}_x + i\hat{\rho}_y] \\
 &= \frac{1}{2l_B^2} \left(\underbrace{[\hat{\rho}_x, \hat{\rho}_x]}_{=0} + \underbrace{[\hat{\rho}_x, i\hat{\rho}_y] + [-i\hat{\rho}_y, \hat{\rho}_x]}_{=2i[\hat{\rho}_x, \hat{\rho}_y]} + \underbrace{[-i\hat{\rho}_y, i\hat{\rho}_y]}_{=0} \right) \\
 &= \frac{-2i^2 l_B^2}{2l_B^2} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 [\hat{a}, \hat{b}^\dagger] &= [\hat{X} + i\hat{Y}, \hat{\rho}_x + i\hat{\rho}_y] \\
 &= [\hat{X}, \hat{\rho}_x] + i[\hat{X}, \hat{\rho}_y] + i[\hat{Y}, \hat{\rho}_x] - [\hat{Y}, \hat{\rho}_y]
 \end{aligned}$$

As $[\hat{X}, \hat{\rho}_x] = [\hat{Y}, \hat{\rho}_y]$ are trivially zero (they contain only operators that commute) we get:

$$\begin{aligned}
 -i[\hat{a}, \hat{b}^\dagger] &= [\hat{X}, \hat{\rho}_y] + [\hat{Y}, \hat{\rho}_x] \\
 &= [\hat{X}, \hat{\rho}_y] + [\hat{Y}, \hat{\rho}_x]
 \end{aligned}$$

The relevant commutators are:

$$\begin{aligned}
 [\hat{X}, \hat{\rho}_y] &= -\frac{1}{4}[\hat{x}, \hat{y}] + \frac{1}{2m\omega}[\hat{x}, \hat{p}_x] + \frac{1}{2m\omega}[\hat{p}_y, \hat{y}] - \frac{1}{m^2\omega^2}[\hat{p}_x, \hat{p}_y] \\
 &= \frac{1}{2m\omega}([\hat{x}, \hat{p}_x] - [\hat{y}, \hat{p}_y]) \\
 &= 0 \\
 [\hat{Y}, \hat{\rho}_x] &= \frac{1}{2m\omega}([\hat{y}, \hat{p}_y] - [\hat{x}, \hat{p}_x]) \\
 &= 0
 \end{aligned}$$

such that

$$-i[\hat{a}, \hat{b}^\dagger] = 0$$

And similarly $[\hat{a}, \hat{b}] = [\hat{a}^\dagger, \hat{b}^\dagger] = 0$.

This means that the operators $\hat{a}, \hat{a}^\dagger, \hat{b}, \hat{b}^\dagger$ follow the same algebra (and the same physics) as two independent harmonic oscillators.

e) The hamiltonian is:

$$H = \frac{1}{2m} (\mathbf{p} - a\mathbf{A})^2 = \frac{1}{2} m \mathbf{v}^2$$

We found the velocity in (2), and by using the result (3), we see we get:

$$\begin{aligned} H &= \frac{1}{2} m (\tilde{\omega} \times (\mathbf{r} - \mathbf{R}))^2 \\ &= \frac{1}{2} m (-\tilde{\omega} \times \tilde{\rho})^2 \\ &\stackrel{\tilde{\omega} \perp \tilde{\rho}}{=} -\frac{1}{2} m \omega^2 \rho^2 \\ &= -\frac{1}{2} m \omega^2 (\rho_x^2 + \rho_y^2) \end{aligned} \quad (5)$$

Expressing these in terms of the ladder operators \hat{b} and \hat{b}^\dagger yields:

$$\hat{\rho}_x = \frac{l_B}{\sqrt{2}} (\hat{b} + \hat{b}^\dagger), \quad \hat{\rho}_y = -\frac{i l_B}{\sqrt{2}} (\hat{b} - \hat{b}^\dagger) \quad (6)$$

$$\begin{aligned} \hat{H} &= \frac{1}{2} m \omega^2 \frac{l_B^2}{2} \left((\hat{b} + \hat{b}^\dagger)^2 - (\hat{b} - \hat{b}^\dagger)^2 \right) \\ &= \frac{1}{4} m \omega^2 l_B^2 (2\hat{b}\hat{b}^\dagger + 2\hat{b}^\dagger\hat{b}) \\ &= \frac{1}{4} m \omega^2 l_B^2 (2[1 + \hat{b}^\dagger\hat{b}] + 2\hat{b}^\dagger\hat{b}) \\ &= m \omega^2 l_B^2 \left(\hat{b}^\dagger\hat{b} + \frac{1}{2} \right) \\ &= m \omega^2 \frac{\hbar}{m |\omega|} \left(\hat{b}^\dagger\hat{b} + \frac{1}{2} \right) \\ &= \hbar \omega \left(\hat{b}^\dagger\hat{b} + \frac{1}{2} \right) \end{aligned} \quad (7)$$

This is the hamiltonian for the harmonic oscillator, and has the energy spectrum $E_n = \hbar \omega (n + \frac{1}{2})$ independent of m . This means that for each energy there are m degenerate states. The angular momentum operator is:

$$L = m (xv_y - yv_x) + \frac{eB}{2} r^2 = m \mathbf{r} \times \mathbf{v} + \frac{eB}{2} \mathbf{r}^2$$

From earlier, we had $\mathbf{r} = \mathbf{R} - \vec{\rho}$, $\mathbf{v} = -\vec{\omega} \times \vec{\rho}$. Then:

$$\begin{aligned} L &= m(\mathbf{R} - \vec{\rho}) \times (-\vec{\omega} \times \vec{\rho}) + \frac{eB}{2} (\mathbf{R} - \vec{\rho})^2 \\ &= m \left(-\vec{\omega} [(\mathbf{R} - \vec{\rho}) \cdot \vec{\rho}] - \vec{\rho} \left[\underbrace{(\mathbf{R} - \vec{\rho}) \cdot (-\vec{\omega})}_{=0} \right] \right) + \frac{eB}{2} (\mathbf{R} - \vec{\rho})^2 \\ &= -m\vec{\omega} (\vec{\rho} \cdot \mathbf{R} - \rho^2) + \frac{eB}{2} (\mathbf{R}^2 - 2\mathbf{R}\vec{\rho} + \rho^2) \\ &= m\vec{\omega} (\rho^2 - \vec{\rho} \cdot \mathbf{R}) - m\omega \left(\frac{\mathbf{R}^2}{2} - \vec{\rho} \cdot \mathbf{R} + \frac{\rho^2}{2} \right) \end{aligned}$$

Since L is zero except for the z component, we can drop the vector notation and have

$$L = \frac{1}{2}m\omega (\rho^2 - \mathbf{R}^2)$$

Quantizing this, and remembering from (5) and (7), we have:

$$\begin{aligned} \rho^2 &= \frac{l_B^2}{2} \left((\hat{b} + \hat{b}^\dagger)^2 - (\hat{b} - \hat{b}^\dagger)^2 \right) \\ &= \frac{l_B^2}{2} \left(2[1 + \hat{b}^\dagger \hat{b}] + 2\hat{b}^\dagger \hat{b} \right) \\ &= \frac{l_B^2}{2} (4\hat{b}^\dagger \hat{b} + 2) \\ &= 2l_B^2 \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \end{aligned}$$

Then, for \mathbf{R} , we can write X and Y in terms of ladder operators as:

$$\hat{X} = \frac{l_B}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{Y} = -\frac{il_B}{\sqrt{2}} (\hat{a} - \hat{a}^\dagger) \quad (8)$$

Which will result in by the same calculation as above in:

$$\mathbf{R}^2 = 2l_B^2 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)$$

Then:

$$\hat{L} = m\omega l_B^2 (\hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a}) = \hbar (\hat{b}^\dagger \hat{b} - \hat{a}^\dagger \hat{a})$$

with eigenvalues:

$$l_{mn} = \hbar(n - m)$$

- f) We want to calculate the expectation values $\langle z | \hat{x} | z \rangle$ and $\langle z | \hat{y} | z \rangle$ for the coherent degenerate ground-state $|z\rangle$ that fullfills $\hat{a}|z\rangle = z|z\rangle$ and $\hat{b}|z\rangle = 0$. From $\hat{\mathbf{r}} = \mathbf{R} - \hat{\rho}$, we get:

$$\hat{x} = \hat{X} - \hat{\rho}_x, \quad \hat{y} = \hat{Y} - \hat{\rho}_y$$

Using the expressions in (6) and (8):

$$\begin{aligned} \langle z|\hat{x}|z\rangle &= \langle z|\hat{X}|z\rangle - \langle z|\hat{\rho}_x|z\rangle \\ &= \langle z|\frac{l_B}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)|z\rangle - \langle z|\frac{l_B}{\sqrt{2}}(\hat{b} + \hat{b}^\dagger)|z\rangle \\ &= \frac{l_B}{\sqrt{2}}\left(\langle z|\hat{a}|z\rangle + \langle z|\hat{a}^\dagger|z\rangle - \langle z|\hat{b}|z\rangle + \langle z|\hat{b}^\dagger|z\rangle\right) \end{aligned}$$

The trick, is to let the hermitian conjugated operators act on the bras, and the regular operators on the kets such that $\hat{a}|z\rangle = z|z\rangle$, $\langle z|\hat{a}^\dagger = (\hat{a}|z\rangle)^\dagger = \langle z|z^*$ and $\hat{b}|z\rangle = 0$. We get:

$$\begin{aligned} \langle z|\hat{x}|z\rangle &= \frac{l_B}{\sqrt{2}}(\langle z|z|z\rangle + \langle z|z^*|z\rangle) \\ &= \frac{l_B}{\sqrt{2}}(z + z^*) \\ &= \sqrt{2}l_B \operatorname{Re}(z) \end{aligned}$$

Onto the next:

$$\begin{aligned} \langle z|\hat{y}|z\rangle &= \langle z|\hat{Y}|z\rangle - \langle z|\hat{\rho}_y|z\rangle \\ &= -\frac{il_B}{\sqrt{2}}\left(\langle z|\hat{a}|z\rangle - \langle z|\hat{a}^\dagger|z\rangle - \underbrace{\langle z|\hat{b}|z\rangle}_{=0} + \underbrace{\langle z|\hat{b}^\dagger|z\rangle}_{=0}\right) \\ &= -\frac{il_B}{\sqrt{2}}(z - z^*) \\ &= \sqrt{2}l_B \operatorname{Im}(z) \end{aligned}$$

Writing $|z\rangle$ in the $|m\rangle$ basis gives:

$$|z\rangle = \sum_m |m\rangle \langle m|z\rangle = e^{-\frac{1}{2}|z|^2} \sum_m \frac{z^m}{\sqrt{m!}} |m\rangle$$

This is gotten by use of equation 1.216 in the lecture notes. When considering how many states that fit in $|z\rangle$ in the lowest landau level for a give z , we use :

$$\langle \hat{x} \rangle^2 + \langle \hat{y} \rangle^2 = \langle \hat{r} \rangle^2 = 2l_B^2 \left(\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2 \right) = 2l_B^2 |z|^2$$

This corresponds to a circle in the complex plane with radius $\sqrt{2}l_B |z|$ and an area

$$\begin{aligned} A &= \pi \left(\sqrt{2}l_B |z| \right)^2 \\ &= 2\pi l_B^2 |z|^2 \end{aligned}$$

The state $|z\rangle$, corresponding to the edge of said circle, has an overlap with the $|m\rangle$ state :

$$\begin{aligned} |\langle m|z\rangle|^2 &= \left| e^{-\frac{1}{2}|z|^2} \sum_{m'} \frac{z^{m'}}{\sqrt{m'!}} \langle m|m'\rangle \right|^2 \\ &= e^{-|z|^2} \frac{z^{2m}}{m!} \\ &= e^{-|z|^2} \frac{(|z|^2)^m}{m!} \end{aligned} \tag{9}$$

To find the m state with maximum overlap with $|z\rangle$ we find:

$$\begin{aligned} \frac{d}{d|z|^2} |\langle m|z\rangle|^2 &= \frac{d}{d|z|^2} \left(e^{-|z|^2} \frac{(|z|^2)^m}{m!} \right) \\ &= -e^{-|z|^2} \frac{(|z|^2)^m}{m!} + e^{-|z|^2} \frac{m(|z|^2)^{m-1}}{m!} \\ &= \frac{e^{-|z|^2} (|z|^2)^{m-1}}{m!} (-|z|^2 + m) \end{aligned}$$

Then we get that $m = |z|^2$. Since the overlap falls exponentially, we can up to a good approximation take the state with $|z|^2$ to be in the pure state m , which means that if we restrict the available space to a circle with radius $r^2 = 2l_B^2|z|^2$ in the z plane we have that $2l_B^2|z|^2 \leq r^2$ and we can restrict m to $2l_B^2m \leq r^2$. As the area is proportional to r^2 we see that the number of states increases linearly with the available area in the complex plane. The density is

$$\rho = \frac{N}{A} = \frac{m}{2\pi l_B^2 m} = \frac{1}{2\pi l_B^2}$$

g) When introducing the electric field, we get an energy contribution:

$$H_E = -e\vec{E} \cdot \vec{r} = -eE x$$

Quantizing this and using the relation $\hat{x} = \hat{X} - \hat{\rho}_x$, we get:

$$\hat{H}_E = -eE (\hat{X} - \hat{\rho}_x) = -\frac{l_B}{\sqrt{2}} eE (\hat{a} + \hat{a}^\dagger - \hat{b} - \hat{b}^\dagger)$$

Then the total hamiltonian is:

$$\hat{H} = \hat{H}_0 + \hat{H}_E = \hbar\omega \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right) - \frac{l_B}{\sqrt{2}} eE (\hat{a} + \hat{a}^\dagger - \hat{b} - \hat{b}^\dagger)$$

In order to only consider the lowest Landau level, i.e $|m, 0\rangle \equiv |m\rangle$, we need the “effective” hamiltonian for this level:

$$\hat{H}|m, 0\rangle = \hbar\omega \left(\underbrace{\hat{b}^\dagger \hat{b}}_{=0} |m, 0\rangle + \frac{1}{2} |m, 0\rangle \right) - \frac{l_B}{\sqrt{2}} eE \left(\hat{a} |m, 0\rangle + \hat{a}^\dagger |m, 0\rangle - \underbrace{\hat{b} |m, 0\rangle}_{=0} - \underbrace{\hat{b}^\dagger |m, 0\rangle}_{=\sqrt{0+1}|m, 1\rangle} \right)$$

We see that the last term isn't in the lowest Landau level, thus, we may neglect it. We're left with:

$$\begin{aligned} \hat{H}' |m, 0\rangle &= \frac{1}{2} \hbar\omega |m, 0\rangle - \frac{l_B}{\sqrt{2}} eE (\hat{a} + \hat{a}^\dagger) |m, 0\rangle \\ \hat{H}' &= \frac{1}{2} \hbar\omega - \frac{l_B}{\sqrt{2}} eE (\hat{a} + \hat{a}^\dagger) \end{aligned}$$

In the Heisenberg picture we can find the time evolution of \hat{a} and \hat{a}^\dagger and get:

$$\begin{aligned} \hat{a}(t)|z\rangle &= \hat{U}(t, 0) \hat{a} \hat{U}(0, t) |z\rangle \\ &= \hat{U}(t, 0) \hat{a} |z(t)\rangle \\ &= \hat{U}(t, 0) z(t) |z(t)\rangle \\ &= z(t) |z\rangle \end{aligned}$$

So:

$$\begin{aligned}\hat{a}(t) &= \hat{U}(t,0)\hat{a}\hat{U}(0,t) \\ &= e^{-it\hat{H}'}\hat{a}e^{it\hat{H}'} \\ &= \hat{a} + \frac{it}{\hbar} [\hat{H}', \hat{a}] + \frac{1}{2!} \left(\frac{it}{\hbar}\right)^2 [\hat{H}', [\hat{H}', \hat{a}]] + \dots\end{aligned}$$

Calculating the commutator:

$$\begin{aligned}[\hat{H}', \hat{a}] &= \left[\frac{1}{2}\hbar\omega - \frac{l_B}{\sqrt{2}}eE(\hat{a} + \hat{a}^\dagger), \hat{a} \right] \\ &= -\frac{l_B}{\sqrt{2}}eE [\hat{a} + \hat{a}^\dagger, \hat{a}] \\ &= -\frac{l_B}{\sqrt{2}}eE \left(\underbrace{[\hat{a}, \hat{a}]}_{=0} + \underbrace{[\hat{a}^\dagger, \hat{a}]}_{=-1} \right) \\ &= \frac{l_B}{\sqrt{2}}eE\end{aligned}$$

Since all operators commute with a scalar, and the “higher order” commutators vanish:

$$\hat{a}(t) = \hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE$$

Then:

$$\begin{aligned}\hat{a}(t)|z\rangle &= \left(z + \frac{itl_B}{\sqrt{2}}eE \right) |z\rangle \\ z(t) &= z + \frac{itl_B}{\sqrt{2}}eE\end{aligned}$$

which shows that $|z\rangle$ gets a time dependence. In order to show that this corresponds to a drift in the y -direction, let's consider:

$$\hat{X}(t) = \frac{l_B}{\sqrt{2}}(\hat{a}(t) + \hat{a}^\dagger(t)), \quad \text{and} \quad \hat{Y}(t) = -\frac{il_B}{\sqrt{2}}(\hat{a}(t) - \hat{a}^\dagger(t))$$

Where $\hat{a}^\dagger(t)$ is:

$$\begin{aligned}(\hat{a}(t))^\dagger &= \left(\hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE \right)^\dagger \\ \hat{a}^\dagger(t) &= \hat{a}^\dagger - \frac{itl_B}{\hbar\sqrt{2}}eE\end{aligned}$$

Then:

$$\hat{X}(t) = \frac{l_B}{\sqrt{2}}(\hat{a}(t) + \hat{a}^\dagger(t)) = \frac{l_B}{\sqrt{2}}\left(\hat{a} + \frac{itl_B}{\hbar\sqrt{2}}eE + \hat{a}^\dagger - \frac{itl_B}{\hbar\sqrt{2}}eE\right) = \hat{X}$$

No drift in the \hat{X} direction, onto \hat{Y} :

$$\begin{aligned}\hat{Y}(t) &= -\frac{il_B}{\sqrt{2}} \left(\hat{a} + \frac{itl_B}{\hbar\sqrt{2}} eE - \hat{a}^\dagger + \frac{itl_B}{\hbar\sqrt{2}} eE \right) \\ &= \hat{Y}(0) + \frac{l_B^2}{\hbar} eEt = \hat{Y}(0) + \frac{e}{|eB|} Et = \hat{Y}(0) - \frac{E}{|B|} t\end{aligned}$$

Then the velocity is:

$$v_{\text{drift}} = -\frac{E}{|B|}$$

in the y direction, as $\hat{Y}(t)$ is the movement of the guiding center (center of orbit).