Problem set 8

8.1 Uncle Charlie's gift

As we have seen in this course, quantum physics can make some unexpected twists to what we normally consider as possible in the communication between two parties. The present problem is due to Jan Myrheim (NTNU).

The eccentric Uncle Charlie has declared his intention to give either his niece Alice (A) or his nephew Bob (B) a generous gift. He has informed them about this and also that the gift is either a million dollars or a new bicycle. In order to test their quantum physics abilities he has sent them one qubit each (by decoherence protected airmail), and has informed them that the two-qubit system is in one out of four possible states,

$$\begin{aligned}
|Aa\rangle &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) \\
|Ab\rangle &= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle) \\
|Ba\rangle &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle) \\
|Bb\rangle &= \frac{1}{2}(-|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned} \tag{1}$$

with $|00\rangle = |0\rangle \otimes |0\rangle$ etc., where the first factor corrsponds to the qubit sent to Alice and the second factor to the qubit sent to Bob. The state of the two qubits contains the information about his decision, with Aa meaning that Alice will get one million dollars, Ab meaning she will get a bicycle, and with similar outcome for Bob when Ba or Bb has been chosen.

Charlie challenges them to find the information by making measurements on their qubits, but Alice and Bob are living far apart, Alice in Norway and Bob in Australia, and their communication is therefore restricted to a classical channel (telephone line) when they want to discuss how to perform the measurement.

After a discussion they reach the frustrating conclusion that they cannot obtain the full information about Uncle Charlie's decision, and they consider instead what is the best information they will be able to extract. The challenge for you is to make a similar analysis.

a) Alice and Bob first consider measuring the qubit states in the $\{|0\rangle, |1\rangle\}$ basis, but they decide that this will give them no information what so ever about Charlie's decision. Why is that the case?

b) At the next step Alice finds that it is better that she measures her qubit in the basis $\{|u\rangle, |v\rangle\}$, where

$$|u\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,, \quad |v\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2}$$

while Bob makes the measurement in the original $\{|0\rangle, |1\rangle\}$ basis. Show that in such a measurement, the four possible outcomes of the measurements would give them the information restricted to the two possibilities, 1: Aa or Ba, 2: Ab or Bb. That means they will get the information about what the gift is but not about who will get this gift.

c) They also consider a measurement where Alice uses the $\{|0\rangle, |1\rangle\}$ basis, while Bob uses the $\{|u\rangle, |v\rangle\}$ basis. What is the information they can get in this way? They further consider the situation here both of them make the measurements in the $\{|u\rangle, |v\rangle\}$ basis. Can more information be extracted with this choice of measurements?

They now make a more complete analysis of the possible measurements by assuming Alice uses a general, orthogonal basis for her measurement,

$$|w\rangle = \alpha|0\rangle + \beta|1\rangle\,, \quad |x\rangle = -\beta^*|0\rangle + \alpha^*|1\rangle \tag{3}$$

with $|\alpha|^2 + |\beta|^2 = 1$, and Bob uses the basis

$$|y\rangle = \gamma|0\rangle + \delta|1\rangle\,, \quad |z\rangle = -\delta^*|0\rangle + \gamma^*|1\rangle \tag{4}$$

with $|\gamma|^2 + |\delta|^2 = 1$.

d) Show that by properly choosing the parameters α, β, γ and δ they will be able to extract the information restricted to the two possibilities, 1: Aa or Bb, 2: Ab or Ba. In this case the result 1 would then tell that either Alice gets a million dollars or Bob gets a bicycle and result 2 would tell them that either Bob gets a million dollars or Alice gets a bicycle. At the end they decide that this may be the most interesting information.

e) Explain why any of the measurements discussed above would erase the rest of the information from the qubits, so that a second measurement would not give any additional information about the gift.

Uncle Charlie's gift would in any case seem unfair and leave at least one of the two discontent. Let us hope that when he realizes that they both have obtained a good understanding of quantum physics through their studies he will compensate in some way the one that does not get the gift in such a way that they both will be happy with the situation.

8.2 Distributed information (Exam 2012)

A secret message is distributed to a party of three, denoted A, B, and C, in the form of an entangled three-spin state, coded into three spin-half particles. As the receiving party knows in advance, the quantum state is one out of a selection of three,

$$|\psi_n\rangle = \frac{1}{\sqrt{3}}(|+--\rangle + \eta^n|-+-\rangle + (\eta^*)^n|--+\rangle)\,, \quad \eta = e^{2\pi i/3} \tag{5}$$

where $n = 0, 1, 2$. The message is identified by the value of n, which means which of the three quantum states that is distributed.

We use the notation $|+--\rangle = |+\rangle\otimes|-\rangle\otimes|-\rangle$ etc., where the single spin states $|\pm\rangle$ are orthogonal states in a basis referred to as *basis I*. The three spinning particles are distributed to A, B and C, one particle to each of them, with the the first state in the tensor product corresponding to the spin sent to A, the second one to B and the third one to C. We assume the three-spin state is preserved under this distribution.

Each person in the receiving party can make (spin) measurements on the spinning particle he/she receives. The three can also communicate over a classical channel, which means that they can correlate their measurements and also compare the results of the measurements. They have, however no quantum channel available for communication. This means that all the observables that are available for measurements by the receiving party are of product form.

a) Determine the reduced density operator of A, and explain why, for any measurement he/she performs on his particle, no information can be extracted about which of the three spin states $|\psi_n\rangle$ is distributed. Also show that if A, B and C all make their spin measurements in *basis I*, even if they communicate their measured results, these cannot make any distinction between the three values of n.

Next, consider the situation where A and B are not able to communicate with C. They decide to perform measurements on the two spins they have received, and to make a probabilistic evaluation for the different values of n, based on the measured results. In order to do so they decide both to make their spin measurements in a rotated basis, which we refer to as *basis II*. The vectors in this basis are

$$|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\,, \quad |1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{6}$$

The possible outcomes of the measurements they list with numbers $k = 1, 2, 3, 4$, with the correspondence

$$k = 1 :\ (0\,,0)\,, \quad k = 2 :\ (0\,,1)\,, \quad k = 3 :\ (1\,,0)\,, \quad k = 4 :\ (1\,,1) \tag{7}$$

We refer to the corresponding states as $|\phi_k\rangle$, with $|\phi_1\rangle = |00\rangle = |0\rangle \otimes |0\rangle$, etc.

Before they do the measurements they evaluate for each three-spin state $|\psi_n\rangle$ the probabilities for the different measurement results (labeled by k). These probabilities are referred to as $p(k|n)$.

b) Find the reduced density operator $\hat{\rho}_n^{AB}$ and determine the probabilities $p(k|n)$ for different values of k and n. It is sufficient, due to repetitions of results, to consider $n = 0, 1$ and $k = 1, 2$. Do you, in particular, see a reason why the probabilities are the same for $n = 1$ and $n = 2$, for all k?

c) Assume now that A and B perform their measurements, with the result labeled by k. The probability for the state to be $|\psi_n\rangle$, under the condition that the measured result is k, we denote by $\bar{p}(n|k)$. Under the assumption that all spin states $|\psi_n\rangle$ are equally probable until the result of the measurement is known, statistics theory gives us the following relation

$$\bar{p}(n|k) = \frac{p(k|n)}{p(k)} \tag{8}$$

with $p(k)$ as a normalization factor. Determine $p(k)$ and the probability $\bar{p}(n|k)$ for each n in the case $k = 1 :\ (0\,,0)$. What message is in this case most probably the one that has been distributed?

8.3 Time evolution in a two-level system (Exam 2013)

The Hamiltonian of a two-level system (denoted A) is $\hat{H}_0 = (1/2)\hbar\omega\,\sigma_z$, with σ_z as the diagonal Pauli matrix. We refer to the normalized ground state vector as $|g\rangle$ and the exited state as $|e\rangle$. In reality the system is coupled to a radiation field (denoted S), and the excited state will therefore decay to the ground state under emission of a quantum of radiation. $\hat{\rho}$ denotes the reduced density operator of subsystem A. To a good approximation the time evolution of this system is described by the Lindblad equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\,[H_0, \hat{\rho}] - \frac{1}{2}\gamma\left[\hat{\alpha}^\dagger\hat{\alpha}\hat{\rho} + \hat{\rho}\hat{\alpha}^\dagger\hat{\alpha} - 2\hat{\alpha}\hat{\rho}\hat{\alpha}^\dagger\right] \tag{9}$$

with γ as the decay rate for the transition $|e\rangle \to |g\rangle$, $\hat{\alpha} = |g\rangle\langle e|$ and $\hat{\alpha}^\dagger = |e\rangle\langle g|$.

In matrix form, with $\{|e\rangle, |g\rangle\}$ as basis, we write the density matrix as $\hat{\rho}$

$$\hat{\rho} = \begin{pmatrix} p_e & b \\ b^* & p_g \end{pmatrix} \tag{10}$$

with p_e as the probability for the system to be in state $|e\rangle$ and p_g as the probability for the system to be in state $|g\rangle$.

a) Assume initially the two-level system, at time $t = 0$, to be in state $\hat{\rho} = |e\rangle\langle e|$. Show, by use of Eq. (9), that p_e decays exponentially, with γ as decay rate, while the total probability $p_e + p_g$ is conserved.

b) Assume next that the system is initially in the following superposition of the two eigenstates of $\hat{H}_0$, $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$. Determine the time dependent density matrix $\hat{\rho}(t)$ with this initial state.

c) The density operator of subsystem A can alternatively be expressed in terms of the Pauli matrices as $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})$. Determine the function $r^2(t)$ in the two cases above and show that in both cases it has a minimum for $t = (1/\gamma)\ln 2$. What is the minimum value for r in the two cases? Comment on the implication the results give for the entanglement between the two subsystems A and S. (We assume A+S all the time to be in a pure state.)