Fermi golden rule $\quad w_{fi} = \frac{W_{fi}}{T} = \frac{2\pi}{\hbar} |\underbrace{\langle f|}_{\langle B, 1_{ka}|} H_{int} \underbrace{|i\rangle}_{|A,0\rangle}|^2 \,\delta(E_f - E_i)$

$$w_{BA} = \sum_{ka} w_{fi} = \frac{2\pi}{\hbar}\sum_{ka} |\langle B, 1_{ka}| H_{int}|A,0\rangle|^2 \,\delta(E_B + \hbar\omega_k - E_A)$$

w_{BA} is time-independent $\qquad W_{BA} = w_{BA}\, t \qquad$ valid if $t \ll \frac{1}{w_{BA}}$

Do not account for the fact that $P_A(t)$ decreases with time.

$$\frac{dW_{BA}}{dt} = \underbrace{w_{BA}}_{\text{FGR}} \cdot P_A(t) \qquad\qquad P_A(t) = 1 - \sum_B W_{BA}$$

$$\frac{dP_A}{dt} = -\sum_B \frac{dW_{BA}}{dt} = -\underbrace{\sum_B w_{BA}}_{\frac{1}{\tau_A}} P_A = -\frac{1}{\tau_A}P_A \quad \overset{P_A(0)=1}{\Longrightarrow} \quad P_A(t) = e^{-t/\tau_A}$$

We can account for this by $\quad |A\rangle \to e^{-t/2\tau_A}|A\rangle$

$$\langle f | U_{int}^I | i\rangle = \langle f|i\rangle - \frac{i}{\hbar}\langle f|H_{int}|i\rangle \int_{t_i}^{t_f} dt\, e^{\frac{i}{\hbar}(E_f-E_i)t}\, e^{-\frac{t-t_i}{2\tau_A}} + \dots$$

$$\overset{\langle f|i\rangle=0}{=} -\frac{i}{\hbar}\langle f|H_{int}|i\rangle\, e^{t_i/2\tau_A}\int_{t_i}^{t_f} dt\, e^{\frac{i}{\hbar}(E_f-E_i+\frac{i\hbar}{2\tau_A})t} + \dots$$

$$= -\frac{\langle f|H_{int}|i\rangle}{E_f-E_i+\frac{i\hbar}{2\tau_A}}\, e^{t_i/2\tau_A}\underbrace{\left[e^{\frac{i}{\hbar}(E_f-E_i+\frac{i\hbar}{2\tau_A})t_f} - e^{\frac{i}{\hbar}(E_f-E_i+\frac{i\hbar}{2\tau_A})t_i}\right]}_{\left[e^{\frac{i}{\hbar}(E_f-E_i)t_f}\underbrace{e^{-\frac{1}{2\tau_A}(t_f-t_i)}}_{\to 0 \quad t_f-t_i \gg \tau_A} - e^{\frac{i}{\hbar}(E_f-E_i)t_i}\right]}$$

$$w_{fi} = \frac{|\langle f|H_{int}|i\rangle|^2}{|\underbrace{E_f - E_i}_{} + \frac{i\hbar}{2\tau_A}|^2} = \frac{|\langle f|H_{int}|i\rangle|^2}{(\hbar\omega_{fi} + \frac{i\hbar}{2\tau_A})(\hbar\omega_{fi} - \frac{i\hbar}{2\tau_A})}$$

$$E_B + \hbar\omega_k - E_A = \hbar\omega_{fi}$$

$$= \frac{|\langle f|H_{int}|i\rangle|^2}{(\hbar\omega_{fi})^2 + \frac{\hbar^2}{4\tau_A^2}}$$