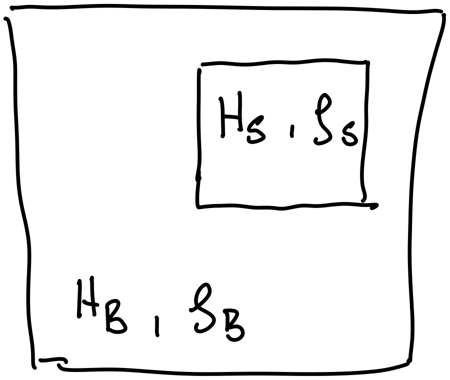

$$H = H_S + H_B + H_I$$

$$\rho$$

$$\begin{tikzcd}
\rho(0) = \rho_S(0) \otimes \rho_B \arrow[r, "U(t)"] \arrow[d, "tr_B"] & \rho(t) = U \rho(0) U^+ \arrow[d, "tr_B"] \\
\rho_S(0) \arrow[r, "\text{Find this}"] & \rho_S(t)
\end{tikzcd}$$

$$H = \underbrace{H_S + H_B}_{H_0} + H_I$$

$$U_0 = e^{-\frac{i}{\hbar} H_0 t} \qquad |\psi_I\rangle = U_0^+ |\psi_S\rangle \qquad \rho^I = U_0^+ \rho U_0$$

$$\frac{d\rho_I}{dt} = -i\left[H_I^I, \rho^I\right] \qquad H_I^I = U_0^+ H_I U_0$$

$$\rho^I(t) = \rho^I(0) - i \int_0^t dt' \left[H_I^I, \rho^I\right]$$

$$\frac{d\rho_I}{dt} = -i\left[H_I^I, \rho^I(0)\right] - \int_0^t dt' \left[H_I^I, \left[H_I^I, \rho^I\right]\right]$$

$$\rho_S^I = Tr_B\, \rho^I$$

$$\frac{d\rho_S^I}{dt} = -i \underbrace{Tr_B\left[H_I^I, \rho^I(0)\right]}_{\to 0} - \int_0^t dt'\, Tr_B\left[H_I^I, \left[H_I^I, \rho^I\right]\right]$$

Born approximation: $\rho(0) = \rho_S(0) \otimes \rho_B(0) \xrightarrow{t} \rho(t) \approx \rho_S(t) \otimes \rho_B(0)$

$$\frac{d\rho_S^I}{dt} = -\int_0^t dt'\, Tr_B\left[H_I^I, \left[H_I^I, \rho_S^I \otimes \rho_B\right]\right]$$

$$H_I = \sum_\alpha S_\alpha \otimes E_\alpha \qquad H_I^I = U_0^+ H_I U_0 = \sum_\alpha \underbrace{e^{iH_S t} S_\alpha e^{-iH_S t}}_{S_\alpha(t)} \otimes \underbrace{e^{iH_B t} E_\alpha e^{-iH_B t}}_{E_\alpha(t)}$$

$$= \sum_\alpha S_\alpha(t) \otimes E_\alpha(t)$$

$$\frac{d\rho_S^I}{dt} = -\int_0^t dt' \sum_{\alpha\beta} \mathrm{Tr}_B\left[S_\alpha(t)\otimes E_\alpha(t), \left[S_\beta(t')\otimes E_\beta(t'), \rho_S^I(t')\otimes\rho_B \right]\right]$$

Correlation function $C_{\alpha\beta}(t,t') = \mathrm{Tr}_B\left[E_\alpha(t)E_\beta(t')\rho_B\right] = \langle E_\alpha(t)E_\beta(t')\rangle_{\rho_B}$

Assume: $[H_B, \rho_B] = 0$ $\quad \rho_B$ is time-independent

$$C_{\alpha\beta}(t,t') = \mathrm{Tr}_B\Big[\underbrace{e^{iH_Bt}}_{e^{iH_B(t-t')}} E_\alpha \underbrace{e^{-iH_Bt}e^{iH_Bt'}}_{e^{-iH_B(t-t')}} E_\beta e^{-iH_Bt'}\rho_B\Big] = \mathrm{Tr}_B\left[E_\alpha(t-t')E_\beta(0)\rho_B\right]$$
$$= C_{\alpha\beta}(t-t')$$

$$\frac{d\rho_S^I}{dt} = -\int_0^t dt' \sum_{\alpha\beta} \Big\{ S_\alpha(t)S_\beta(t')\rho_S^I(t')\underbrace{\mathrm{Tr}_B\left(E_\alpha(t)E_\beta(t')\rho_B\right)}_{C_{\alpha\beta}(t-t')}$$
$$- S_\alpha(t)\rho_S^I(t')S_\beta(t')\underbrace{\mathrm{Tr}_B\left(E_\alpha(t)\rho_B E_\beta(t')\right)}_{C_{\beta\alpha}(t'-t)}$$
$$- S_\beta(t')\rho_S^I(t')S_\alpha(t)\underbrace{\mathrm{Tr}_B\left(E_\beta(t')\rho_B E_\alpha(t)\right)}_{C_{\alpha\beta}(t-t')}$$
$$+ \rho_S^I(t')S_\beta(t')S_\alpha(t)\underbrace{\mathrm{Tr}_B\left(\rho_B E_\beta(t')E_\alpha(t)\right)}_{C_{\beta\alpha}(t'-t)} \Big\}$$

$$\frac{d\rho_S^I}{dt} = -\int_0^t dt' \sum_{\alpha\beta} \Big\{ C_{\alpha\beta}(t-t')\left[S_\alpha(t)S_\beta(t')\rho_S^I(t') - S_\beta(t')\rho_S^I(t')S_\alpha(t)\right]$$
$$+ C_{\beta\alpha}(t'-t)\left[\quad - - - - \quad \right]\Big\}$$

$C_{\alpha\beta}(t-t')$ is small for $t-t' > \tau_R$

Markov approx: inside $\int dt'$ $\quad \rho_S^I(t') \to \rho_S^I(t)$ $\quad t-t' = \tau$

$$\frac{d\rho_S^I}{dt} = -\int_0^{t \to \infty} d\tau \sum_{\alpha\beta} \Big\{ C_{\alpha\beta}(\tau)\left[S_\alpha(t)S_\beta(t-\tau)\rho_S^I(t) - S_\beta(t-\tau)\rho_S^I(t)S_\alpha(t)\right]$$
$$+ C_{\beta\alpha}(-\tau)\left[\; - \; - \; — \; \right]\Big\}$$

Define $$S_\beta(\omega) = \sum_{\varepsilon'-\varepsilon=\omega} |\varepsilon\rangle\langle\varepsilon| S_\beta |\varepsilon'\rangle\langle\varepsilon'| \qquad (\text{S.P.})$$

$$H_S|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$$

$$\sum_\omega S_\beta(\omega) = \underbrace{\sum_\omega \sum_{\varepsilon'-\varepsilon=\omega}}_{\sum_{\varepsilon,\varepsilon'}} |\varepsilon\rangle\langle\varepsilon| S_\beta |\varepsilon'\rangle\langle\varepsilon'| = S_\beta$$

$$[H_S, S_\beta(\omega)] = \underbrace{H_S S_\beta(\omega)}_{\sum_{\varepsilon'-\varepsilon=\omega} \varepsilon |\varepsilon\rangle\langle\varepsilon| S_\beta |\varepsilon'\rangle\langle\varepsilon'|} - S_\beta(\omega) H_S = -\sum_{\varepsilon'-\varepsilon=\omega} (\overbrace{\varepsilon'-\varepsilon}^{\omega}) |\varepsilon\rangle\langle\varepsilon| S_\beta |\varepsilon'\rangle\langle\varepsilon'|$$

$$= -\omega S_\beta(\omega)$$

I.P.
$$S_\beta^I(\omega,t) = e^{iH_S t} S_\beta(\omega) e^{-iH_S t} = e^{-i\omega t} S_\beta(\omega)$$

$$e^{iB} A e^{-iB} = \cdots$$

$$\frac{d\rho_S^I}{dt} = -\int_0^\infty d\tau \sum_{\substack{\alpha\beta \\ \omega,\omega'}} \Big\{ C_{\alpha\beta}(\tau) \Big[e^{-i\omega t} S_\alpha(\omega) e^{i\omega'(t-\tau)} S_\beta^\dagger(\omega') \rho_S^I(t)$$
$$- S_\beta^\dagger(\omega') \rho_S^I(t) S_\alpha(\omega) e^{i(\omega'-\omega)t} e^{-i\omega'\tau} \Big]$$
$$+ C_{\beta\alpha}(-\tau) [\cdots] \Big\}$$

$$= -\sum_{\substack{\alpha\beta \\ \omega\omega'}} \Big\{ e^{i(\omega'-\omega)t} \underbrace{\int_0^\infty d\tau\, C_{\alpha\beta}(\tau) e^{-i\omega'\tau}}_{\Gamma_{\alpha\beta}(\omega')} \Big[S_\alpha(\omega) S_\beta^\dagger(\omega') \rho_S^I(t) - S_\beta^\dagger(\omega') \rho_S^I(t) S_\alpha(\omega) \Big] + C_{\beta\alpha} [--] \Big\}$$

If $H_I \ll H_S$ $\quad \rho_S^I(t)$ changes little on the time scale $\frac{1}{\omega'-\omega}$

$\Rightarrow$ Assume $\omega' = \omega$ (approx)

$$\frac{d\rho_S^I}{dt} = -\sum_{\substack{\alpha\beta \\ \omega}} \Big\{ \Gamma_{\alpha\beta}(\omega) \Big[S_\alpha(\omega) S_\beta^\dagger(\omega) \rho_S^I(t) - S_\beta^\dagger(\omega) \rho_S^I(t) S_\alpha(\omega) \Big] + \Gamma_{\beta\alpha}^* [--] \Big\}$$

$$\Gamma_{\alpha\beta}(\omega) = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + i\, S_{\alpha\beta}(\omega) \qquad \gamma_{\alpha\beta}(\omega) = \Gamma_{\alpha\beta}(\omega) + \Gamma^{*}_{\beta\alpha}(\omega)$$

$$S_{\alpha\beta}(\omega) = \frac{1}{2i}\left(\Gamma_{\alpha\beta}(\omega) - \Gamma^{*}_{\beta\alpha}(\omega)\right)$$

$$\frac{d\rho_S^I}{dt} = -i\left[H_{LS}, \rho_S^I(t)\right] - \frac{1}{2}\sum_{\substack{\alpha\beta\\ \omega}} \gamma_{\alpha\beta}(\omega)\Big(S_\alpha(\omega) S_\beta^\dagger(\omega)\rho_S^I + \rho_S^I S_\alpha(\omega) S_\beta^\dagger(\omega) - 2 S_\beta^\dagger(\omega)\rho_S^I S_\alpha(\omega)\Big)$$

$$H_{LS} = \sum_{\alpha,\beta,\omega} S_{\alpha\beta}(\omega) S_\beta^\dagger(\omega) S_\alpha(\omega)$$

$$D = \begin{pmatrix}\gamma_1 & & 0\\ & \gamma_2 & \\ 0 & & \ddots\end{pmatrix}$$

Diagonalize $\gamma_{\alpha\beta}(\omega) = M_{\alpha\gamma} D_{\gamma\delta} M^\dagger_{\delta\beta}$ $\qquad L_\gamma(\omega) = M_{\alpha\gamma}(\omega) S_\alpha(\omega)$

$$\sum_{\alpha\beta\omega} \gamma_{\alpha\beta}(\omega) S_\alpha(\omega) S_\beta^\dagger(\omega) = \sum_{\alpha,\omega} \gamma_\alpha(\omega) L_\alpha(\omega) L_\alpha^\dagger(\omega)$$

Back to S.P.

$$\frac{d\rho_S}{dt} = -i\left[H, \rho_S\right] - \frac{1}{2}\sum_{\alpha,\omega}\gamma_\alpha(\omega)\Big[L_\alpha(\omega) L_\alpha^\dagger(\omega)\rho_S + \rho_S L_\alpha(\omega) L_\alpha^\dagger(\omega) - 2 L_\alpha^\dagger(\omega)\rho_S L_\alpha(\omega)\Big]$$

Lindblad eq.