Harmonic oscillator: $H = \hbar\omega a^+ a$

$|n\rangle \qquad H|n\rangle = E_n|n\rangle \qquad E_n = n\hbar\omega$

Lindblad op. $L = a$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}\left(a^+ a\rho + \rho a^+ a - 2a\rho a^+\right)$$

$\gamma_1 : |1\rangle \to |0\rangle \qquad \gamma_2 : |2\rangle \to |1\rangle \quad \ldots \qquad \gamma = \gamma_1 = \gamma_2 = \ldots$

$\rho_{mn} = \langle m|\rho|n\rangle$

$$\langle m|[H,\rho]|n\rangle = \underbrace{\langle m|H}_{E_m\langle m|}\rho - \rho\underbrace{H|n\rangle}_{E_n|n\rangle} = (E_m - E_n)\rho_{mn}$$

$$\langle m|\underbrace{a^+a}_{m}\rho + \rho\underbrace{a^+a}_{n} - 2a\rho a^+|n\rangle = (m+n)\rho_{mn} - 2\sqrt{m+1}\sqrt{n+1}\,\rho_{m+1,n+1}$$

$$\sqrt{m+1}\langle m+1| \qquad \sqrt{n+1}|n+1\rangle$$

$$\frac{d\rho_{mn}}{dt} = -\frac{i}{\hbar}\rho_{mn}(E_m - E_n) - \frac{\gamma}{2}(m+n)\rho_{mn} + \gamma\sqrt{m+1}\sqrt{n+1}\,\rho_{m+1,n+1}$$

$$P_n = \rho_{nn} \qquad \frac{dP_n}{dt} = -\gamma n P_n + \gamma(n+1)P_{n+1}$$

$|n\rangle \to |n-1\rangle$

$w_{BA} = w_{n-1,n} = n\gamma$

$$w_{BA} = \frac{4\alpha}{3c^2}\omega_{BA}^3 |x_{BA}|^2$$

$$x_{BA} = \langle B|\hat{x}|A\rangle \qquad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a^+ + a)$$

$$= \langle n-1|(a^+ + \underbrace{a)|n\rangle}_{\sqrt{n}|n-1\rangle}\sqrt{\frac{\hbar}{2m\omega}}$$

$\sim n \qquad\qquad \sim \sqrt{n}$

$w_{BA} = \gamma n$

Energy relaxation: $\quad E_n = \hat{n}\hbar\omega \qquad n = \langle\hat{n}\rangle = \langle a^+a\rangle = Tr(a^+a\rho)$

$$E = \langle\hat{H}\rangle = \hbar\omega n$$

$$\frac{d\langle n\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}(a^\dagger a\,\rho) = \sum_n \langle n|\underbrace{a^\dagger a}_{n\langle n|}\frac{d\rho}{dt}|n\rangle = \sum_n n\,\frac{dp_n}{dt}$$

$$= \sum_n n\left(-n\gamma\, p_n + (n+1)\gamma\, p_{n+1}\right) = -\gamma \sum_n \Big(n^2 p_n \overbrace{-\,n(n+1)p_{n+1} - (n+1)p_{n+1}}^{-(n+1)^2 p_{n+1}} + (n+1)p_{n+1}\Big)$$

$$= -\gamma \sum_n \Big(\underbrace{n^2 p_n - (n+1)^2 p_{n+1}}_{\sum_n \to 0} + (n+1)\,p_{n+1}\Big) = -\gamma \underbrace{\sum_{n=0}^{\infty}(n+1)\,p_{n+1}}_{\sum_{n=1}^{\infty} n p_n = \sum_{n=0}^{\infty} n p_n = \langle n\rangle} = -\gamma\langle n\rangle$$

$$\frac{d\langle n\rangle}{dt} = -\gamma\langle n\rangle \quad\Rightarrow\quad \langle n\rangle = e^{-\gamma t}\langle n(0)\rangle$$

Initial state $|n\rangle$

$$\frac{d\rho_{mn}}{dt} = -\frac{i}{\hbar}\rho_{mn}(E_m - E_n) - \frac{\gamma}{2}(m+n)\rho_{mn} + \gamma\sqrt{m+1}\sqrt{n+1}\,\rho_{m+1,n+1}$$

$$\rho_{mn} = \begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \cdots \\ \rho_{10} & \rho_{11} & \rho_{12} & \cdots \\ \rho_{20} & \rho_{21} & \rho_{22} & \cdots \\ & & & \end{pmatrix}$$

$$|n\rangle \to \rho = |n\rangle\langle n| \qquad = \begin{array}{c c} & n \\ n & \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 0 & & & & \\ \vdots & & & 1 & \\ & 0 & & & 0 \end{pmatrix} \end{array}$$

$$p_n = \rho_{nn}$$

$$\frac{dp_n}{dt} = -n\gamma\, p_n + (n+1)\gamma\, p_{n+1}$$

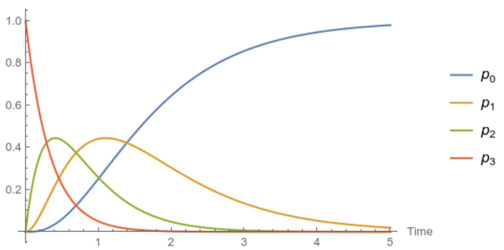
Ex: $|3\rangle \qquad n=3 \qquad p_3(0) = 1 \qquad p_2(0) = p_1(0) = p_0(0) = 0$

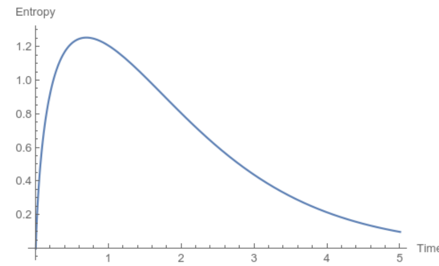
$$\frac{dp_3}{dt} = -3\gamma\, p_3 \quad\Rightarrow\quad p_3(t) = e^{-3\gamma t}$$

$$\frac{dp_2}{dt} = -2\gamma\, p_2 + 3\gamma\, p_3 = -2\gamma p_2 + 3\gamma e^{-3\gamma t} \quad\Rightarrow\quad p_2(t) = 3\left(e^{-2\gamma t} - e^{-3\gamma t}\right)$$

$$\frac{dp_1}{dt} = -\gamma p_1 + 2\gamma p_2 = -\gamma p_1 + 2\gamma\, 3\left(e^{-2\gamma t} - e^{-3\gamma t}\right) \quad\Rightarrow\quad p_1(t) = 3e^{-3\gamma t}\left(e^{\gamma t} - 1\right)^2$$

$$\frac{dp_0}{dt} = +\gamma p_1 = 3\gamma e^{-3\gamma t}(e^{\gamma t}-1)^2 \quad \Rightarrow \quad p_0(t) = e^{-3\gamma t}(e^{\gamma t}-1)^3$$





Start from coherent state: $|z\rangle$ $\quad a|z\rangle = z|z\rangle$

Define $\quad C(\lambda, \lambda^*, t) = \mathrm{Tr}(\rho e^{\lambda a^\dagger} e^{-\lambda^* a})$

$$\frac{\partial C}{\partial \lambda} = \mathrm{Tr}(\rho a^\dagger e^{\lambda a^\dagger} e^{-\lambda^* a})$$

$$\frac{\partial C}{\partial \lambda^*} = -\mathrm{Tr}(a\rho e^{\lambda a^\dagger} e^{-\lambda^* a})$$

$$\frac{\partial C}{\partial t} = \mathrm{Tr}\Big(\underbrace{\frac{\partial \rho}{\partial t}}_{\hookrightarrow -\frac{i}{\hbar}[H,\rho] - \frac{\gamma}{2}(a^\dagger a \rho + \rho a^\dagger a - 2a\rho a^\dagger)} e^{\lambda a^\dagger} e^{-\lambda^* a}\Big)$$

$$= -i\omega \mathrm{Tr}\big((a^\dagger a\rho - \rho a^\dagger a)e^{\lambda a^\dagger} e^{-\lambda^* a}\big) - \frac{\gamma}{2}\mathrm{Tr}\big[(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger)e^{\lambda a^\dagger} e^{-\lambda^* a}\big]$$

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \dots$$

$$e^{\lambda^* a} a^\dagger e^{-\lambda^* a} = a^\dagger + \lambda^* \underbrace{[a,a^\dagger]}_{1} = a^\dagger + \lambda^* \Rightarrow a^\dagger e^{-\lambda^* a} = e^{-\lambda^* a}(a^\dagger + \lambda^*)$$

$$e^{-\lambda a^\dagger} a e^{\lambda a^\dagger} = a - \lambda \underbrace{[a^\dagger, a]}_{-1} = a + \lambda \Rightarrow a e^{\lambda a^\dagger} = e^{\lambda a^\dagger}(a+\lambda)$$

$$\mathrm{Tr}(a^\dagger a\rho e^{\lambda a^\dagger} e^{-\lambda^* a}) = \mathrm{Tr}\big(a\rho(a^\dagger - \lambda^*)e^{\lambda a^\dagger} e^{-\lambda^* a}\big) = \mathrm{Tr}(a\rho a^\dagger e^{\sim} e^{\sim}) + \lambda^* \frac{\partial C}{\partial \lambda^*}$$

$$\mathrm{Tr}(\rho a^\dagger \underbrace{a e^{\lambda a^\dagger}}_{e^{\lambda a^\dagger}(a+\lambda)} e^{-\lambda^* a}) = \mathrm{Tr}\big((a+\lambda)\rho a^\dagger e^{\sim} e^{\sim}\big) = \mathrm{Tr}(a\rho a^\dagger e^{\sim} e^{\sim}) + \lambda \frac{\partial C}{\partial \lambda^*}$$

$$\frac{\partial C}{\partial t} = -\left(\frac{\gamma}{2} + i\omega\right)\lambda^* \frac{\partial C}{\partial \lambda^*} - \left(\frac{\gamma}{2} - i\omega\right)\lambda \frac{\partial C}{\partial \lambda}$$

$$C_0(\lambda, \lambda^*) = C(\lambda, \lambda^*, 0)$$

$$c(\lambda,\lambda^*,t) = c_0\left(\lambda e^{-(\frac{\gamma}{2}-i\omega)t},\ \lambda^* e^{-(\frac{\gamma}{2}+i\omega)t}\right)$$

$t=0$: $\quad |z\rangle$

$$c_0(\lambda,\lambda^*) = \mathrm{Tr}\Big(\underbrace{|z\rangle}_{e^{-\lambda^* z}}\overbrace{\langle z| e^{\lambda a^\dagger}}^{e^{\lambda z^*}} e^{-\lambda^* a}\Big) = e^{\lambda z^* - \lambda^* z}$$

$$c(\lambda,\lambda^*,t) = e^{\lambda \underbrace{e^{-(\frac{\gamma}{2}-i\omega)t} z^*}_{z^*(t)} - \lambda^* \underbrace{e^{-(\frac{\gamma}{2}+i\omega)t} z}_{z(t)}} = e^{\lambda z^*(t) - \lambda^* z(t)}$$

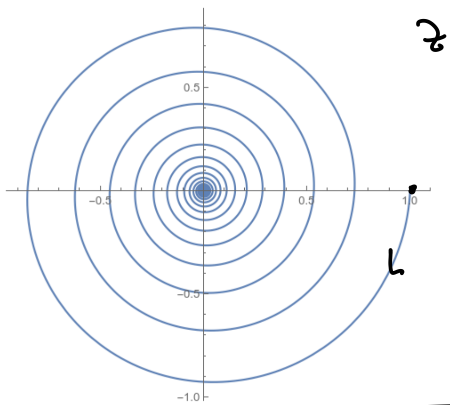
$$z(t) = e^{-(\frac{\gamma}{2}+i\omega)t} z$$

$\Rightarrow \rho(t) = |z(t)\rangle\langle z(t)|$

$|\psi(t)\rangle = |z(t)\rangle$ pure state Entropy $S=0$

$z(t)$ $\quad t\to\infty \quad z(t)\to 0$

$|\psi\rangle \to |0\rangle$



$t=0$: $\quad |\psi\rangle = N\left(|z_1\rangle + |z_2\rangle\right)$

$$\rho(0) = |\psi\rangle\langle\psi| = |N|^2\left(|z_1\rangle\langle z_1| + |z_1\rangle\langle z_2| + |z_2\rangle\langle z_1| + |z_2\rangle\langle z_2|\right)$$

$$c_0(\lambda,\lambda^*) = \mathrm{Tr}\left(\rho(0) e^{\lambda a^\dagger} e^{-\lambda^* a}\right)$$

$$\mathrm{Tr}\Big(\underbrace{|z_1\rangle}_{e^{-\lambda^* z_1}}\underbrace{\langle z_2| e^{\lambda a^\dagger}}_{e^{\lambda z_2}} e^{-\lambda^* a}\Big) = e^{-\lambda^* z_1 + \lambda z_2}\underbrace{\mathrm{Tr}\left(|z_1\rangle\langle z_2|\right)}_{\sum_n \langle n|z_1\rangle\langle z_2|n\rangle \;=\; \langle z_2|n\rangle\langle n|z_1\rangle} = \langle z_2|z_1\rangle e^{-\lambda^* z_1 + \lambda z_2}$$

$$c_0(\lambda,\lambda^*) = |N|^2\left(e^{\lambda z_1^* - \lambda^* z_1} + \langle z_2|z_1\rangle e^{\lambda z_2^* - \lambda^* z_1} + \langle z_1|z_2\rangle e^{\lambda z_1^* - \lambda^* z_2} + e^{\lambda z_2^* - \lambda^* z_2}\right)$$

$$C(\lambda,\lambda^*,t) = |N|^2\left(e^{\lambda z_1^*(t) - \lambda^* z_1(t)} + \langle z_2|z_1\rangle e^{\lambda z_2^*(t) - \lambda^* z_1(t)} + \langle z_1|z_2\rangle e^{\lambda z_1^* - \lambda^* z_2} + e^{\lambda z_2^*(t) - \lambda^* z_2(t)}\right)$$

For state $|\psi(t)\rangle = N(|z_1(t)\rangle + |z_2(t)\rangle)$ (superposition)

$$C^S(\lambda,\lambda^*,t) = \ldots\ldots \langle z_2(t)|z_1(t)\rangle \ldots \langle z_1(t)|z_2(t)\rangle \ldots$$

$$\eta(t) = \frac{\langle z_1|z_2\rangle}{\langle z_1(t)|z_2(t)\rangle} \qquad |\langle z_1|z_2\rangle| = e^{-\frac{1}{2}|z_2-z_1|^2}$$

$$|\eta(t)| = \frac{e^{-\frac{1}{2}|z_2-z_1|^2}}{e^{-\frac{1}{2}|z_2(t)-z_1(t)|^2}} \qquad z_i(t) = e^{-\left(\frac{\gamma}{2}+i\omega\right)t} z_i$$

$$|z_2(t)-z_1(t)|^2 = e^{-\gamma t}|z_2-z_1|$$

$$= e^{-\frac{1}{2}|z_2-z_1|^2(1-e^{-\gamma t})}$$

$\gamma t \ll 1$: $\quad 1-e^{-\gamma t} \approx 1-(1-\gamma t\ldots) = \gamma t + \ldots$

$$|\eta(t)| = e^{-\Gamma t} \qquad \Gamma = \frac{1}{2}\gamma|z_2-z_1|^2$$

if $|z_2-z_1| \gg 1$.

$\Gamma \gg \gamma$

$\Gamma\gamma \gg 1$,

$$C(\lambda,\lambda^*,t) \approx |N|^2\left(e^{\lambda z_1^*(t) - \lambda^* z_1(t)} + 1 \to 2\right)$$

$$\rho(t) \approx |N|^2\left(|z_1(t)\rangle\langle z_1(t)| + |z_2(t)\rangle\langle z_2(t)|\right)$$

<u>Ex</u> $\quad z_1 = 0 \qquad z_2 = z \qquad |\psi(0)\rangle = |N|(|0\rangle + |z\rangle)$

$$\tau = \frac{1}{\Gamma} = \frac{2}{\gamma \underbrace{|z|^2}_{\langle n\rangle}} = \frac{2}{\gamma\langle n\rangle}$$

rate to emit one photon.

$$\omega_{BA} = \omega_{n-1,n} = \gamma n$$